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Kleisli morphisms and randomized congruences for the Giry monad

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Abstract

Stochastic relations are the Kleisli morphisms for the Giry monad. This paper proposes the study of the associated morphisms and congruences. The relationship between kernels of these morphisms and congruences is studied, and a unique factorization of a morphism through this kernel is shown to exist. This study is based on an investigation into countably generated equivalence relations on the space of all subprobabilities. Operations on these relations are investigated quite closely. This utilizes positive convex structures and indicates cross-connections to Eilenberg–Moore algebras for the Giry monad. Hennessy–Milner logic serves as an illustration for randomized morphisms and congruences.
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1. Introduction and motivation

Moggi proposes in his paper *Notions of Computations and Monads* [20] a monadic model of computation. Functor $\mathbb{T}$ acts as a type constructor, so that if A is a type, then $\mathbb{T}A$ is the object of computations of type A. Assuming that $\mathbb{T}$ is the functorial part of a monad, the Kleisli category for this monad is identified as the category of programs, and morphisms in this category are the programs.

The present paper is about the Kleisli morphisms in the Giry monad, a monad that was proposed and investigated by Giry [9] as one component for the categorical foundation of probability theory; see also [23,22]. This monad is computationally interesting due to its close connections to probabilistic testing [18,29] or to its use for modeling software architectures [8,16]. It may be used as the basic structure for stochastic Kripke models for different kinds of modal logics or for continuous time stochastic logics [4,3,5]. This structure is fairly rich, mathematically interesting, and versatile.

Formally, most of these applications are modeled through stochastic relations. Such a stochastic relation $\mathsf{K} = (X, Y, K)$ operates on the measurable spaces X and Y by defining a measurable map $K : X \to \mathbb{S}(Y)$, with $\mathbb{S}$ as the functor which assigns to each measurable space the space of all subprobability measures. This is a coalgebraic view: if $X = Y$, a stochastic relation is perceived as a coalgebra for this functor. A morphism $\mathsf{f} : \mathsf{K}_1 \to \mathsf{K}_2$ is a pair of Borel maps $\mathsf{f} = (\varphi, \psi)$ for which $K_2 \circ \varphi = \mathbb{S}(\psi) \circ K_1$, thus we know that $K_2(\varphi(x_1))(B_2) = K_1(x_1)(\psi^{-1}[B_2])$

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holds, whenever $x_1 \in X_1$ and $B_2 \subseteq Y_2$ is a measurable set. Structurally, however, this idea of a morphism does not take into account that the functor is really the functorial part of a monad, and that this monad defines a Kleisli category. In fact, a stochastic relation is a Kleisli morphism for this monad. Shifting the attention from the measurable map $K : X \to \mathbb{S}(Y)$ to the Kleisli morphism $K : X \rightsquigarrow Y$, suitable morphisms need to be defined: a morphism $\mathsf{F} = (\Phi, \Psi) : \mathsf{K}_1 \curvearrowright \mathsf{K}_2$ in this category is comprised of Kleisli morphisms $\Phi : X_1 \rightsquigarrow X_2$ and $\Psi : Y_1 \rightsquigarrow Y_2$ with $K_2 * \Phi = \Psi * K_1$, where the composition operator $*$ is *Kleisli composition*. These morphisms will be called *randomized* in the present paper.

Related work. Panangaden defines in [22] the category **SRel** of stochastic relations in close analogy to the category of set-valued relations. The objects in this category are measurable spaces, a morphism between two objects is a stochastic relation between them, composition is Kleisli composition. This relation is defined and briefly discussed. In terms of **SRel**, the present paper investigates the category **MSRel** of morphisms on **SRel** that has as objects (X, Y, K) with measurable spaces taken from the subcategory of analytic spaces, and $K : X \rightsquigarrow Y$ a morphism in **SRel**. A morphism in **MSRel** is a pair $(\Phi, \Psi) : (X_1, Y_1, K_1) \to (X_2, Y_2, K_2)$ with $\Phi : X_1 \rightsquigarrow X_2$, $\Psi : Y_1 \rightsquigarrow Y_2$ such that $\Psi * K_1 = K_2 * \Phi$. This is the view put forward above. In [1, Section 7] an interesting special case is discussed: consider as objects pairs (X, μ) with X Polish and μ a probability on the Borel sets of X, then a morphism $\theta : (X, \mu) \to (X', \mu')$ is a probability measure θ on the Cartesian product $X \times X'$ whose respective marginals are absolutely continuous with respect to μ and μ'. Using disintegration, it is shown then that morphisms can be constructed from stochastic relations $K : X \rightsquigarrow X'$; this yields also an approach to forming the converse of a stochastic relation. In [7], stochastic relations are used to investigate bisimulations for labeled Markov transition systems with a view towards investigating logical and behavioral equivalence and bisimilarity, [6] deals with the Giry monad proper and identifies the Eilenberg–Moore algebras for this monad (so to speak the other end of the spectrum, when it comes to identify the adjunctions that give rise to the present monad). The argumentation in this work dealing with stochastic relations stresses that there is a close analogy to set-valued relations, and in fact a careful analysis reveals many striking similarities [8], but also some interesting differences. Taylor [28, Exercises III.53, Section VI.6.3] discusses relational morphisms in the category of relations (and relates this to Osius' work [21] on a categorical foundation of set theory). Neither paper, however, discusses these morphisms in greater detail.

Motivation. The present paper investigates these morphisms more closely, in particular we are interested in factoring a morphism through its kernel. This is a classical algebraic question, cp. [17, p. 104]. It has been answered in the positive for stochastic relations (with conventional, non-randomized morphisms) in [8, Section 5.2]; the discussion of randomized morphisms is apparently new. When dealing with these kernels, one needs to define congruences, in this case on subprobabilities. The interplay of equivalence relations on a base space X and on the corresponding space $\mathbb{S}(X)$ of subprobabilities is observed and investigated. This is necessary because general equivalence relations on $\mathbb{S}(X)$ are far too general to be useful for our purposes. An equivalence relation γ on $\mathbb{S}(X)$ leaves as a trace – as a footprint, so to speak – the equivalence relation $\lfloor \gamma \rfloor$ on X through the monad's unit. Conversely, an equivalence relation α on X can be lifted (or randomized) to become a relation $\overline{\alpha}$ on $\mathbb{S}(X)$. The interplay between a relation γ on $\mathbb{S}(X)$ and the randomization $\overline{\lfloor \gamma \rfloor}$ of its trace on X will be scrutinized, since it soon turns out to be crucial for our purposes. We will require $\gamma \subseteq \overline{\lfloor \gamma \rfloor}$ to hold for obtaining sensible results; this property, that is dubbed *near-grounded*, comes up quite naturally when discussing morphisms. Kernels of morphisms will be required to satisfy a condition related to near-groundedness as well. There is even a condition called groundedness which means that γ is insensitive to lifting its trace, but this turns out too strong a property.

Thus we will first investigate equivalence relations on $\mathbb{S}(X)$ and their relation to equivalences on the base space, in particular we deal with tracing and lifting of countably generated relations. A characterization of grounded relations is provided first for Polish, then for analytic spaces, indicating a somewhat surprising connection to the Eilenberg–Moore algebras for the Giry monad via positive convexity. The reduction technique from analytic to Polish spaces that is developed here is interesting in its own right. Armed with these tools, we investigate randomized congruences for stochastic relations. The interplay between these congruences and their non-randomized cousins is rather interesting as well.

Randomized morphisms are defined in terms of the Kleisli category, they are generalizations of the morphisms studied already in depth for stochastic relations. We concentrate rather quickly in the discussion on the kernels of these morphisms, because we want to investigate the link between them and congruences. This requires – similarly to congruences – finding a good and firm hub of these morphisms between the base space X and the space $\mathbb{S}(X)$ which essentially says that the behavior of kernel equivalent subprobabilities must not be spoiled by their behavior on kernel invariant Borel sets. It is then shown that each randomized morphism of this class can be factored uniquely through its kernel.

Hennessy–Milner logic serves as an illustration to morphisms and congruences. We define randomized morphisms between labeled Markov transitions systems (which are basically stochastic Kripke models). The congruence induced by the logic is shown to be a congruence for these morphisms, providing an analogue to the well-known observation that a formula is valid in a state iff it is valid in its image under a morphism. There we observe an interesting interplay between the theory induced by the logic and issues of measurability through invariant sets; these sets take the form of states that cannot be separated by the logic.

We do not discuss in this paper general measurable spaces, but assume rather that we work in Polish and analytic spaces. These spaces enjoy quite favorable measure-theoretic properties, their measurable structure is rather well known, and they occur quite naturally in applications.

Organization. Section 2 will provide the reader with some background material, making the paper self-contained; we define in particular, stochastic relations and their morphisms, congruences, the Giry monad, etc. Section 3 discusses in greater detail the interplay of equivalence relations on a base space, and on the space of its subprobabilities, it defines grounded and near-grounded equivalence relations and investigates them. This section contains also a result on a Borel isomorphism of certain factor spaces which may be of independent interest. Section 4 deals with randomized morphisms and their kernels, a factorization result is given in Section 6, Section 5 illustrates the development through a simple logic. Finally, Section 7 wraps it all up and gives some suggestions for further work.

2. Preliminaries

This section collects for the reader's convenience some basic facts that will be helpful in the sequel: Polish spaces, smooth equivalence relations, stochastic relations and their morphisms, congruences. A comprehensive treatment of these topics in the context of the algebraic theory of stochastic relations can be found e.g. in [8, Chapter 1].

Polish and analytic spaces. Given measurable spaces $(A, \mathcal{M})$ and $(B, \mathcal{N})$ – thus $\mathcal{M}$ and $\mathcal{N}$ are σ-algebras on A resp. B – a map $f : A \to B$ is called $\mathcal{M}$-$\mathcal{N}$-*measurable* (or simply *measurable*, when the context is clear) whenever $f^{-1}[\mathcal{N}] \subseteq \mathcal{M}$, thus the inverse image $f^{-1}[B]$ of every member B of $\mathcal{N}$ is a member of $\mathcal{M}$.

A Polish space X is a second countable topological space for which a complete metric exists. The Borel sets $\mathcal{B}(X)$ are the smallest σ-algebra on X which contain the open sets of X. Measurability refers always to the Borel sets, unless otherwise specified.

The following observation [27, Corollary 3.2.6] will be most helpful in the sequel.

Lemma 2.1. *Let X be a Polish space with topology $\mathcal{T}$, and $f : X \to Z$ be a measurable map, where Z is a separable metric space with topology $\mathcal{Z}$. Then there exists a Polish topology $\mathcal{T}'$ on X with these properties:*

1. *$\mathcal{T} \subseteq \mathcal{T}'$, thus $\mathcal{T}'$ is finer than $\mathcal{T}$.*
2. *The Borel sets on X with respect to $\mathcal{T}$ and with respect to $\mathcal{T}'$ are the same.*
3. *f is $\mathcal{T}'$-$\mathcal{Z}$-continuous.* □

An *analytic space* is a Hausdorff topological space that is the image of a Polish space under a continuous, or, what amounts to the same, under a Borel map. Hence it makes sense to talk about the Borel sets of an analytic space. Similarly, a set $A \subseteq Y$ of a Polish space Y is called an *analytic set* iff it is the image of a Borel set under a continuous, or – what again amounts to be the same – under a Borel map. The famous Souslin Theorem [27, Theorem 4.4.3] characterizes Borel sets in terms of analytic ones.

Theorem 2.2 (*Souslin*). *Let Y be an analytic space, then a subset A is a Borel set iff both A and $Y \setminus A$ are analytic.* □

Smooth equivalence relations. Let $(A, \mathcal{M})$ be a measurable space, then an equivalence relation ξ is called *smooth* (or *countably generated*) iff there exists a sequence $(Q_n)_{n\in\mathbb{N}}$ of sets in $\mathcal{M}$ such that

$$a \; \xi \; a' \quad \text{iff } \forall n \in \mathbb{N} : [a \in Q_n \Leftrightarrow a' \in Q_n].$$

The sequence $(Q_n)_{n\in\mathbb{N}}$ is said to *determine* ξ.

It is well known (compare [27, Exercise 5.1.10]) that for analytic X a relation ξ is smooth iff there exists a separable measurable space $(Z, \mathcal{Z})$ and a measurable map $f : X \to Z$ such that ξ equals the kernel of f, i.e.

$$\xi = \mathsf{ker}(f) := \{\langle x, x'\rangle \mid f(x) = f(x')\}.$$

Here the measurable space $(Z, \mathcal{Z})$ is called *separable* iff $\mathcal{Z}$ is generated by a sequence $(C_n)_{n\in\mathbb{N}}$ of sets that separate points (given two different points in Z there exists C_n that does contain exactly one of them). Analytic spaces are separable as measurable spaces [15, Proposition 12.1].

Call for an equivalence relation ξ on a set A a set $B \subseteq A$ ξ*-invariant* iff $B = \bigcup\{[b]_\xi \mid b \in B\}$, so that $b \in B$ and $b\,\xi\,b'$ implies $b' \in B$. The ξ-invariant measurable sets $\mathcal{INV}(\mathcal{M}, \xi)$ form a σ-algebra for a measurable space $(A, \mathcal{M})$.

Lemma 2.3. *Let X be an analytic space, and ξ an equivalence relation on X.*

(i) *If ξ is determined by the sequence $(Q_n)_{n\in\mathbb{N}}$ of Borel sets, then*

$$\mathcal{INV}(\mathcal{B}(X), \xi) = \sigma(\{Q_n \mid n \in \mathbb{N}\}).$$

(ii) $x\,\xi\,x'$ *iff* $\forall B \in \mathcal{INV}(\mathcal{B}(X), \xi) : [x \in B \Leftrightarrow x' \in B]$.

Proof. (1) Part (i) follows from [27, Lemma 3.1.6].

(2) Part (ii) is an easy consequence: Define the equivalence relation $x\,\xi_0\,x'$ iff $[x \in B \Leftrightarrow x' \in B]$ for all $B \in \mathcal{INV}(\mathcal{B}(X), \xi)$. $\{Q_n \mid n \in \mathbb{N}\} \subseteq \mathcal{INV}(\mathcal{B}(X), \xi)$ implies $\xi_0 \subseteq \xi$. For establishing the reverse inclusion, fix x, x' with $x\,\xi\,x'$, and put

$$\mathcal{D} := \{B \in \mathcal{INV}(\mathcal{B}(X), \xi) \mid x \in B \Leftrightarrow x' \in B\},$$

then $\mathcal{D}$ is a σ-algebra that contains $\{Q_n \mid n \in \mathbb{N}\}$, consequently, $\mathcal{D} = \mathcal{INV}(\mathcal{B}(X), \xi)$. This implies $\xi \subseteq \xi_0$, and we are done. □

Two remarks are in order. First, the proof technique for the second part will be applied quite frequently, when the equality of σ-algebras needs to be established: one collects the good guys, i.e. the sets for which the property holds, and shows that they form a σ-algebra that contains some generator $\mathcal{D}$, thus must include the σ-algebra $\sigma(\mathcal{D})$ as well. The second remark addresses part (ii): the property of equivalent elements to be either both in a set or neither is inherited from a generator to the σ-algebra. This entails that the invariant sets uniquely determine the equivalence relation.

Let X be analytic, then the factor space X/ξ is analytic, whenever ξ is smooth, provided the factor carries the largest σ-algebra that makes the factor map $\eta_\xi : x \mapsto [x]_\xi$ $\mathcal{B}(X)$-measurable [27, Exercise 5.1.14]. This property renders analytic spaces for our purposes at least as attractive as Polish spaces. Polish spaces are not closed under factoring through smooth relations [2]. An important and helpful consequence of Souslin's Theorem 2.2 is that each ξ-invariant Borel set $A \subseteq X$ can be written as

$$A = \eta_\xi^{-1}\left[\eta_\xi\,[A]\right],$$

and that

$$\mathcal{B}(X/\xi) = \{\eta_\xi\,[A] \mid A \in \mathcal{INV}(\mathcal{B}(X), \xi)\}.$$

Factoring a factor space through a smooth relation will not really bring new structural information: one can show that the iterated factor space is isomorphic to a factor space that is obtained from a relation on the base space, presenting an occasion for introducing a kind of multiplicative operation on relations. Assume that ξ is a smooth equivalence relation on the analytic space X, and that τ is a smooth equivalence on X/ξ. Define for $x, x' \in X$

$$x\ (\tau \bullet \xi)\ x' \Leftrightarrow [x]_\xi\ \tau\ \left[x'\right]_\xi.$$

We obtain [8, Proposition 5.2]:

Lemma 2.4. *The equivalence relation $\tau \bullet \xi$ is smooth, and the analytic spaces $X/\tau \bullet \xi$ and $(X/\xi)/\tau$ are Borel isomorphic.* □

Subprobabilities. Let $\mathbb{S}(A, \mathcal{M})$ denote for a measurable space $(A, \mathcal{M})$ the set of all subprobability measures on $\mathcal{M}$; this set is endowed with the weak*-σ-algebra $\mathcal{M}^\bullet$. This is the smallest σ-algebra that renders the evaluation map $\mu \mapsto \mu(B)$ measurable for each set $B \in \mathcal{M}$. If X is Polish, then $\mathbb{S}(X)$ is a Polish space under the weak topology as well; this is the smallest topology making the maps $\mu \mapsto \int_X f \,\mathrm{d}\mu$ continuous, where $f : X \to \mathbb{R}$ is bounded and continuous [24, Chapter II]. Then $\mathcal{B}(X)^\bullet = \mathcal{B}(\mathbb{S}(X))$ [15, Theorem 17.24] has been discovered many times in the literature on labeled Markov transition systems.

$\mathbb{S}$ is a functor on the category of measurable spaces with measurable maps, where an $\mathcal{M}$-$\mathcal{N}$-measurable map $f : A \to B$ is assigned the map $\mathbb{S}(f) : \mathbb{S}(A) \to \mathbb{S}(B)$ through

$$\mathbb{S}(f)(\mu)(E) := \mu(f^{-1}[E]).$$

The construction yields for each $g : B \to \mathbb{R}$ measurable and bounded

$$\int_B g \,\mathrm{d}\mathbb{S}(f)(\mu) = \int_A g \circ f \,\mathrm{d}\mu.$$

The latter equality is commonly referred to as the *Change of Variables* formula. It is not difficult to see that $\mathbb{S}(f)$ is $\mathcal{B}(X)^\bullet$-$\mathcal{B}(Y)^\bullet$-measurable, provided X is Polish and Y is separable metric. If the Borel map $f : X \to Y$ is onto, so is $\mathbb{S}(f) : \mathbb{S}(X) \to \mathbb{S}(Y)$ [8, Proposition 1.30]. If X is an analytic space, so is $\mathbb{S}(X)$ with $\mathcal{B}(\mathbb{S}(X)) = \mathcal{B}(X)^\bullet$. It is well known that $\mathbb{S}$ is the functorial part of the Giry monad [9].

Definition 2.5. The *Giry monad* $(\mathbb{S}, \mathfrak{e}, \mathfrak{m})$ has the subprobability functor $\mathbb{S}$ on analytic spaces as its functorial part, the monad's *multiplication* is defined through

$$\mathfrak{m}_X(M)(E) := \int_{\mathbb{S}(X)} \tau(E)\, M(\mathrm{d}\tau)$$

$(M \in \mathbb{S}^2(X)$, $E \subseteq X$ measurable), and its *unit* as

$$\mathfrak{e}_A(a)(E) := \delta_a(E),$$

δ_a being the Dirac measure on a.

The Eilenberg–Moore algebras for this monad are exactly the positive convex structures (see Definition 3.13), when the base category are the Polish spaces with continuous maps [8, Section 3.3].

Returning to the general discussion of noteworthy properties of measurable spaces, we note that equality for two measures on a σ-algebra entails equality of the integral for functions that are measurable with respect to that σ-algebra. This will be helpful below.

Lemma 2.6. *Let $\mathcal{A} \subseteq \mathcal{B}(X)$ be a σ-algebra, and assume that $\mu(A) = \mu'(A)$ holds for all $A \in \mathcal{A}$. Then*

$$\int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu'$$

holds for all $f : X \to \mathbb{R}$ that are $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable.

Proof. It is no loss of generality to assume that f is non-negative, for f can be written as $f = f^+ - f^-$ with positive part $f^+ := \max\{f, 0\}$ and negative part $f^- := \max\{-f, 0\}$, and the integral is additive.

Since f is $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable, the set

$$\{f > t\} := \{x \in X \mid f(x) > t\}$$

is a member of $\mathcal{A}$ for every $t \in \mathbb{R}$, thus, using the Choquet integral,

$$\int_X f \,\mathrm{d}\mu = \int_0^\infty \mu(f > t)\mathrm{d}t = \int_0^\infty \mu'(f > t)\mathrm{d}t = \int_X f \,\mathrm{d}\mu'. \quad \square$$

Stochastic relations. Fix measurable spaces $(A, \mathcal{M})$ and $(B, \mathcal{N})$, then $K : (A, \mathcal{M}) \rightsquigarrow (B, \mathcal{N})$ is a *stochastic relation* iff $K : A \to \mathbb{S}(B, \mathcal{N})$ is a measurable map; in Probability Theory, a stochastic relation would be called a sub-Markov kernel or a transition sub-probability. We will call $\mathsf{K} = (X, Y, K)$ *Polish* or *analytic* depending on whether both X and Y are Polish resp. analytic spaces. A relation without a qualification does not make any assumption at all on its underlying measurable spaces. We have the following characterization:

Proposition 2.7. *Let $(A, \mathcal{M})$ and $(B, \mathcal{M})$ be measurable spaces. Then the following statements are equivalent:*

(i) $K : A \to \mathbb{S}(B, \mathcal{N})$ *is* $\mathcal{M}$-$\mathcal{N}^\bullet$*-measurable.*
(ii) $K : A \to (\mathcal{N} \to [0, 1])$ *is a map such that*
  a. $K(a)$ *is a subprobability on* $\mathcal{N}$ *for all* $a \in A$.
  b. $a \mapsto K(a)(E)$ *is an* $\mathcal{M}$*-measurable map for each* $E \in \mathcal{N}$.
(iii) K *is a morphism in the Kleisli category associated with the Giry monad.* □

This characterization will be used silently throughout. The *Kleisli product* $L * K$ of two stochastic relations $K : A \rightsquigarrow B$ and $L : B \rightsquigarrow C$ is characterized through ($a \in A$, $E \subseteq C$ measurable)

$$(L * K)(a)(E) = \int_B L(b)(E)\ K(a)(\mathrm{d}b).$$

It corresponds to the composition of morphisms in the Kleisli category associated with the monad; see [19, Section VI.5] for a general account.

Morphisms and congruences. Let $\mathsf{K} = (X, Y, K)$ be an analytic relation. The pair $\mathsf{c} = (\rho, \tau)$ of smooth equivalence relations on X resp. Y is called a *congruence* for K iff $K(x)(B) = K(x')(B)$ holds, whenever $x\ \rho\ x'$ and $B \in \mathcal{INV}(\mathcal{B}(Y), \tau)$ is a τ-invariant subset of Y. Thus K behaves in the same way for inputs from X that cannot be separated by ρ and outputs from Y that cannot be separated by τ.

Congruences can be described in terms of measurability with respect to the invariant sets for the associated equivalence relations.

Lemma 2.8. *Let $\mathsf{K} = (X, Y, K)$ be an analytic relation, and assume that $\mathsf{c} = (\rho, \tau)$ is a pair of smooth equivalence relations on X resp. Y. Then these statements are equivalent*

(i) c *is a congruence for* K.
(ii) $K : (X, \mathcal{INV}(\mathcal{B}(X), \rho)) \rightsquigarrow (Y, \mathcal{INV}(\mathcal{B}(Y), \tau))$ *is a stochastic relation.*

Proof. (1) For '(i) ⇒ (ii)', it is sufficient to show that $x \mapsto K(x)(B)$ is $\mathcal{INV}(\mathcal{B}(X), \rho)$-$\mathcal{B}(\mathbb{R})$-measurable, whenever $B \subseteq Y$ is a τ-invariant Borel set in Y. To this end, it suffices to show that $\{x \in X \mid K(x)(B) < t\}$ is a member of $\mathcal{INV}(\mathcal{B}(X), \rho)$ for each $t \in \mathbb{R}$. Since K is a stochastic relation, the latter set is a Borel set, since c is a congruence, this set is ρ-invariant.

(2) For '(ii) ⇒ (i)', let t be a real number, take x_1, x_2 with $x_1\ \rho\ x_2$, and assume that $B \in \mathcal{INV}(\mathcal{B}(Y), \tau)$ is a τ-invariant Borel set in Y. Because $\{x \in X \mid K(x)(B) \geq t\} \in \mathcal{INV}(\mathcal{B}(X), \rho)$, we have $K(x_1)(B) \geq t$ iff $K(x_2)(B) \geq t$, thus $K(x_1)(B) = K(x_2)(B)$. This implies that c is a congruence. □

If $\mathsf{K} = (X, Y, K)$ and $\mathsf{L} = (A, B, L)$ are both general stochastic relations, then $\mathsf{f} = (\varphi, \psi) : \mathsf{K} \to \mathsf{L}$ is a *morphism* iff $\varphi : X \to A$ and $\psi : Y \to B$ are surjective measurable maps such that $L \circ \varphi = \mathbb{S}(\psi) \circ K$ holds, hence iff the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "K"'] & A \arrow[d, "L"] \\
\mathbb{S}(Y) \arrow[r, "\mathbb{S}(\psi)"'] & \mathbb{S}(B)
\end{tikzcd}$$

commutes. Spelling the condition out, it means that $L(\varphi(x))(E) = K(x)(\psi^{-1}[E])$ holds for each $x \in X$ and each measurable $E \subseteq B$.

We will need to factor stochastic relations through analytic relations; the result will be a stochastic relation again. To be specific [8, Section 5.2]:

Proposition 2.9. *Let* $\mathsf{K} = (X, Y, K)$ *be a stochastic relation over the analytic spaces* X *and* Y, *and assume that* $\mathsf{c} = (\alpha, \beta)$ *is a congruence for* K.

(i) *Define*

$$(K/\mathsf{c})\,([x]_\alpha)(B) := K(x)(\eta_\beta^{-1}[B]),$$

then $K/\mathsf{c} : X/\alpha \rightsquigarrow Y/\beta$ *is a stochastic relation.*

(ii) $\eta_{\mathsf{c}} := (\eta_\alpha, \eta_\beta) : \mathsf{K} \to \mathsf{K}/\mathsf{c}$ *is a morphism.*

(iii) *If* $\mathsf{c}' = (\alpha', \beta')$ *is a congruence for* K *with* $\alpha \subseteq \alpha'$ *and* $\beta \subseteq \beta'$, *then there exists a unique congruence* $\mathsf{d} = (\xi, \zeta)$ *for* K/c *with* $\mathsf{c}' = \mathsf{d} \bullet \mathsf{c} := (\xi \bullet \alpha, \zeta \bullet \beta)$.

(iv) *If* d *is a congruence for* K/c, *then there exists a unique morphism* $g : K/\mathsf{c} \to K/\mathsf{d} \bullet \mathsf{c}$ *with* $\eta_{\mathsf{d}\bullet\mathsf{c}} = g \circ \eta_{\mathsf{c}}$.

$\square$

The fact that each analytic space is the image of a Polish space under a Borel map is extended to stochastic relations: given an analytic stochastic relation L, we can construct a Polish relation K and a morphism $\mathsf{f} : \mathsf{K} \to \mathsf{L}$, see [8, Lemma 4.2].

3. Grounded relations on subprobabilities

We will study the interplay of equivalence relations on $\mathbb{S}(X)$ and X, resp. by investigating the trace that an equivalence relation ξ on $\mathbb{S}(X)$ leaves on the base space X, and by lifting (or randomizing) an equivalence relation on X to $\mathbb{S}(X)$. The trace is constructed through the unit for the Giry monad, and lifting is done through comparing measures on invariant sets. Both approaches seem to be fairly natural when it comes to transform an equivalence relation from one space to another. The question arises whether constructing the trace of a lifted equivalence relation yields the original relation, and whether lifting the trace does change anything. These questions will present themselves when randomized morphisms are considered, so we deal with them here in suitable generality and at considerable length.

The central rôle is played in the present discussions by the invariant sets for a smooth equivalence relation. On the one hand, they are defined on the base space, on the other hand they are part of the domain of a subprobability measure, hence they permit comparing these measures. Thus they serve as a kind of hub in the discussion that will follow. We will often take an equivalence relation on $\mathbb{S}(X)$ and see what we can say about the invariant sets with respect to its trace. Thus a relation of the subprobabilities is interlocked with those on the base space.

We begin with the randomization of an equivalence relation on a measurable space, and with the converse process, a kind of de-randomization, which has a look at the trace on the base space. We will compare the relations that arise, putting particular emphasis on those relations on $\mathbb{S}(X)$ that are finer than the randomization of their trace, and on those which equal this randomization; we call the latter ones grounded. A complete characterization of grounded relations can be given in terms of positive convex structures, hinting at a connection between these relations and the Eilenberg–Moore algebras for the Giry monad.

Definition 3.1. Given a measurable space $(A, \mathcal{M})$ and an equivalence relation ξ on A, define on $\mathbb{S}(A, \mathcal{M})$ the *lifting* $\overline{\xi}$ *of* ξ through $\mu \,\overline{\xi}\, \mu'$ iff $\mu(B) = \mu'(B)$ for all $B \in \mathcal{INV}(\mathcal{M}, \xi)$.

Thus the subprobabilities μ and μ' are $\overline{\xi}$-equivalent iff they behave in the same way on the ξ-invariant sets in $\mathcal{M}$. The randomization of an equivalence relation on A is clearly an equivalence relation on $\mathbb{S}(A, \mathcal{M})$.

Let X be an analytic space with Borel sets $\mathcal{B}(X)$, and let α be smooth; we will fix X and α throughout.

An elementary consequence of Lemma 2.3 is that we can recover α from $\overline{\alpha}$.

Corollary 3.2. $x \;\alpha\; x'$ *iff* $\delta_x \;\overline{\alpha}\; \delta_{x'}$. $\square$

Here are two examples.

Example 3.3. Let X be an analytic space.

(i) The identity Δ_X is a smooth equivalence relation, and $\mathcal{INV}(\mathcal{B}(X), \Delta_X)$ is easily established to equal $\mathcal{B}(X)$. Thus $\overline{\Delta_X} = \Delta_{\mathbb{S}(X)}$.

(ii) The universal relation $U_X = X \times X$ is smooth with $\mathcal{INV}(\mathcal{B}(X), U_X) = \{\emptyset, X\}$. Thus $\overline{U_X} = \{\langle \mu, \mu' \rangle \mid \mu(X) = \mu'(X)\}$. $\square$

Lemma 3.4. *If $\alpha = \mathsf{ker}(f)$ for the Borel map $f : X \to T$ and the analytic space T, then $\overline{\alpha} = \mathsf{ker}(\mathbb{S}(f))$. Hence $\overline{\alpha}$ is a smooth equivalence relation on $\mathbb{S}(X)$.*

Proof. (0) It is no loss of generality to assume that f is onto, since the image of an analytic set under a Borel map is again an analytic set [27, Exercise 4.1.3].

(1) Note that $f^{-1}[B]$ is α-invariant for $B \in \mathcal{B}(T)$. Thus $\langle \mu, \mu' \rangle \in \mathsf{ker}(\mathbb{S}(f))$ implies

$$\mu(f^{-1}[B]) = \mathbb{S}(f)(\mu)(B) = \mathbb{S}(f)(\mu')(B) = \mu'(f^{-1}[B])$$

for all Borel sets $B \in \mathcal{B}(T)$. Because each invariant Borel set $A \in \mathcal{INV}(\mathcal{B}(X), \alpha)$ may be represented as the inverse image under f of a Borel set in T, we see that $\langle \mu, \mu' \rangle \in \overline{\alpha}$.

(2) Conversely, let $\mu(A) = \mu'(A)$ for each $A \in \mathcal{INV}(\mathcal{B}(X), \alpha) = f^{-1}[\mathcal{B}(T)]$, thus

$$\mathbb{S}(f)(\mu)(B) = \mu(f^{-1}[B]) = \mu'(f^{-1}[B]) = \mathbb{S}(f)(\mu')(B)$$

is established for each Borel set $B \in \mathcal{B}(T)$. This in turn implies that the pair $\langle \mu, \mu' \rangle$ is in the kernel of $\mathbb{S}(f)$. $\square$

Conversely, assume that γ is a smooth equivalence relation on $\mathbb{S}(X)$. We ask under which conditions there exists a smooth equivalence relation α on X such that $\gamma = \overline{\alpha}$. Define the equivalence relation $\lfloor \gamma \rfloor$ on X upon setting

$$x \lfloor \gamma \rfloor x' \Leftrightarrow \delta_x \, \gamma \, \delta_{x'}.$$

Then $\lfloor \gamma \rfloor$ is smooth: if $\gamma = \mathsf{ker}(H)$, then $\lfloor \gamma \rfloor = \mathsf{ker}(H \circ \mathfrak{e}_X)$, where $\mathfrak{e}_X : X \ni x \mapsto \delta_x \in \mathbb{S}(X)$ maps each point into its Dirac measure.

We are interested in those smooth relations on $\mathbb{S}(X)$ that are related to their traces on X, so they deserve a special name.

Definition 3.5. A smooth equivalence relation γ on $\mathbb{S}(X)$ is called

1. *near-grounded* iff $\gamma \subseteq \overline{\lfloor \gamma \rfloor}$,
2. *grounded* iff $\gamma = \overline{\lfloor \gamma \rfloor}$.

Suppose that γ is near-grounded, then $\mu \, \gamma \, \mu'$ implies $\mu(B) = \mu'(B)$ for all Borel sets that are $\lfloor \gamma \rfloor$-invariant; in this way γ, which is defined on $\mathbb{S}(X)$, is related to the $\lfloor \gamma \rfloor$-invariant Borel sets of X. Hence we have at least some hints at how μ and μ' behave on the base set.

Constructing $\overline{\lfloor \cdot \rfloor}$ is a bit akin to forming the closure of the interior of a set in a topological space. If γ is grounded, then it is uniquely determined by its restriction on $\{\delta_x \mid x \in X\}$, thus essentially by its behavior on the base set. This is intuitively clear and will be formally substantiated in Proposition 3.14.

We give an example for a grounded equivalence relation, and provide an example indicating that there are near-grounded relations which are not grounded.

Example 3.6. Let $X := [-1, 1]$ and put

$$\mu \, \gamma \, \mu' \Leftrightarrow \forall A \in \mathcal{B}(X) : \int_A t^2 \, \mu(\mathrm{d}t) = \int_A t^2 \, \mu'(\mathrm{d}t),$$

then γ is a smooth equivalence relation. In fact, let $\mathcal{A}$ be a countable generator for $\mathcal{B}(X)$, and define

$$H(\mu) := \left(\int_A t^2 \, \mu(\mathrm{d}t) \right)_{A \in \mathcal{A}},$$

then $H : \mathbb{S}(X) \to [0, 1]^{\mathbb{N}}$ is continuous with $\gamma = \mathsf{ker}(H)$.

It is clear that $x_1 \lfloor \gamma \rfloor x_2$ iff $x_1^2 = x_2^2$, so that a Borel set $S \in \mathcal{B}(X)$ is $\lfloor \gamma \rfloor$-invariant iff S is symmetric, thus iff $r \in S$ implies $-r \in S$. Consequently,

$$\mu \, \overline{\lfloor \gamma \rfloor} \, \mu' \Leftrightarrow \int_S t^2 \, \mu(\mathrm{d}t) = \int_S t^2 \, \mu'(\mathrm{d}t) \quad \text{for all symmetric Borel sets } S.$$

This implies that $\gamma = \overline{\lfloor \gamma \rfloor}$, so γ is grounded. $\square$

Example 3.7. Let $K : X \rightsquigarrow Y$ be a stochastic relation with X, Y Polish, and define for $\mu \in \mathcal{B}(X)$ and $E \in \mathcal{B}(X \times Y)$

$$(\mu \otimes K)(E) := \int_X K(x)(E_x)\, \mu(\mathrm{d}x),$$

where $E_x := \{y \in Y \mid \langle x, y\rangle \in E\}$ is the vertical cut of E at x. Put

$$\mu\ \gamma_K\ \mu' \Leftrightarrow \mu \otimes K = \mu' \otimes K$$

then γ_K is smooth on account of $\mu \mapsto \mu \otimes K$ defining a Borel map $\mathbb{S}(X) \to \mathbb{S}(X \times Y)$.

Then $\lfloor \gamma_K \rfloor = \mathsf{ker}(K)$. Now let $K = \nu$ be constant, hence $\mu \otimes K$ equals the product measure $\mu \otimes \nu$. Furthermore $\mathsf{ker}(K) = X \times X$, so that $\mathcal{INV}(\mathcal{B}(X), \lfloor \gamma_\nu \rfloor) = \{\emptyset, X\}$, see Example 3.3. Consequently, $\mu\ \overline{\lfloor \gamma_\nu \rfloor}\ \mu'$ iff $\mu(X) = \mu'(X)$. Now select μ, μ', ν so that $\mu(X) = \mu'(X)$ but $\mu \otimes \nu \neq \mu' \otimes \nu$, indicating $\overline{\lfloor \gamma_\nu \rfloor} \not\subseteq \gamma_\nu$. Thus γ_ν is near-grounded but not grounded. □

3.1. A Borel isomorphism

The dependence of a grounded equivalence relation γ on its trace $\lfloor \gamma \rfloor$ on the base space is strong, because in this case γ is completely determined by $\lfloor \gamma \rfloor$. We will show that this is reflected on the Borel structure of its factor space $\mathbb{S}(X)/\gamma$, which equals $\mathbb{S}(X/\lfloor \gamma \rfloor)$ up to a Borel isomorphism. Thus this factor is essentially a space of probability measures.

It turns out that the construction under consideration can be carried out already for a near-grounded relation, yielding a surjective Borel map $\partial_\gamma : \mathbb{S}(X)/\gamma \to \mathbb{S}(X/\lfloor \gamma \rfloor)$. Grounded relations are different from their near-grounded cousins: this is witnessed by the fact that γ is grounded iff $\mathbb{S}(X)/\gamma$ and $\mathbb{S}(X/\lfloor \gamma \rfloor)$ are isomorphic. We study in this section the map ∂_γ and its properties, providing some insight into the inner working of the transformations between equivalence relations on different, albeit closely related spaces.

Define for a smooth and near-grounded equivalence relation γ on $\mathbb{S}(X)$ with X analytic the map

$$\partial_\gamma : \begin{cases} \mathbb{S}(X)/\gamma \to \mathbb{S}(X/\lfloor \gamma \rfloor) \\ [\mu]_\gamma \mapsto \mathbb{S}(\eta_{\lfloor \gamma \rfloor})(\mu). \end{cases}$$

Thus we have

$$\partial_\gamma([\mu]_\gamma)(C) = \mu(\eta_{\lfloor \gamma \rfloor}^{-1}[C])$$

for the Borel set $C \in \mathcal{B}(X/\lfloor \gamma \rfloor)$ and $\mu \in \mathbb{S}(X)$.

Lemma 3.8. *Whenever X is analytic, and γ is a smooth and near-grounded equivalence relation on $\mathbb{S}(X)$, ∂_γ is a surjective and $\mathcal{B}(\mathbb{S}(X)/\gamma)$-$\mathcal{B}(\mathbb{S}(X/\lfloor \gamma \rfloor))$-measurable map.*

Proof. (0) Suppose $\mu\ \gamma\ \mu'$, then $\mu(B) = \mu'(B)$ whenever $B \in \mathcal{INV}(\mathcal{B}(X), \lfloor \gamma \rfloor)$, since γ is near-grounded. Thus $\mathbb{S}(\eta_{\lfloor \gamma \rfloor})(\mu) = \mathbb{S}(\eta_{\lfloor \gamma \rfloor})(\mu')$, consequently, ∂_γ is well-defined. Given $\mu \in \mathbb{S}(X)$, it is clear that $\partial_\gamma([\mu]_\gamma) \in \mathbb{S}(X/\lfloor \gamma \rfloor)$.

(1) Let $\nu \in \mathbb{S}(X/\lfloor \gamma \rfloor)$, then there exists $\mu \in \mathbb{S}(X)$ such that $\nu = \mathbb{S}(\eta_{\lfloor \gamma \rfloor})(\mu)$ [8, Proposition 1.30]. Thus $\partial_\gamma([\mu]_\gamma) = \nu$.

(2) In order to establish measurability of ∂_γ, we need to show that $\partial_\gamma^{-1}[G]$ is a Borel set in $\mathbb{S}(X)/\gamma$, whenever $G \subseteq \mathbb{S}(X/\lfloor \gamma \rfloor)$ is a Borel set. Since the Borel sets of $\mathbb{S}(X/\lfloor \gamma \rfloor)$ are exactly the elements of the weak*-σ-algebra with respect to the Borel sets on $X/\lfloor \gamma \rfloor$, we may assume that there exist $H \in \mathcal{B}(X/\lfloor \gamma \rfloor)$ and a Borel set $F \subseteq \mathbb{R}$ such that

$$G = \{\tau \in \mathbb{S}(X/\lfloor \gamma \rfloor) \mid \tau(H) \in F\}.$$

Because $H \in \mathcal{B}(X/\lfloor \gamma \rfloor)$, we know that $\eta_{\lfloor \gamma \rfloor}^{-1}[H] \in \mathcal{INV}(\mathcal{B}(X), \lfloor \gamma \rfloor)$, thus we want to show that

$$\partial_\gamma^{-1}[G] = \{[\mu]_\gamma \mid \mu(\eta_{\lfloor \gamma \rfloor}^{-1}[H]) \in F\} \in \mathcal{B}(\mathbb{S}(X)/\gamma),$$

equivalently, that $\eta_\gamma^{-1}[\partial_\gamma^{-1}[G]]$ is an γ-invariant member of $\mathcal{B}(\mathbb{S}(X))$. Now

$$\eta_\gamma^{-1}[\partial_\gamma^{-1}[G]] = \left(\partial_\gamma \circ \eta_\gamma\right)^{-1}[G] = \{\mu \in \mathbb{S}(X) \mid \mu(\eta_{\lfloor \gamma \rfloor}^{-1}[H]) \in F\},$$

and this is a γ-invariant Borel set in $\mathbb{S}(X)$. □

If γ is grounded, we can even say a bit more:

Lemma 3.9. *If γ is a near-grounded equivalence relation on $\mathbb{S}(X)$ with X analytic, then ∂_γ is a bijection iff γ is grounded.*

Proof. (1) Take $\mu_1, \mu_2 \in \mathbb{S}(X)$ which lie in different γ-classes, then there exists for grounded γ an $\lfloor\gamma\rfloor$-invariant Borel set $B \in \mathcal{INV}(\mathcal{B}(X), \lfloor\gamma\rfloor)$ such that $\mu_1(B) \neq \mu_2(B)$, consequently, because

$$C = \eta_{\lfloor\gamma\rfloor}^{-1}\left[\eta_{\lfloor\gamma\rfloor}[C]\right]$$

holds for each $\lfloor\gamma\rfloor$-invariant Borel set C, we obtain

$$\partial_\gamma([\mu_1]_\gamma)(\eta_{\lfloor\gamma\rfloor}[B]) = \mu_1(B) \neq \mu_2(B) = \partial_\gamma([\mu_2]_\gamma)(\eta_{\lfloor\gamma\rfloor}[B]).$$

(2) Conversely, it is easy to see that γ is grounded if ∂_γ is one-to-one. □

It will be shown now, moreover, that ∂_γ is a Borel isomorphism, and not only a Borel measurable, bijective map. This requires some preparatory work. We claim that the image of certain Borel sets under $\mathbb{S}(\eta_{\lfloor\gamma\rfloor})$ are Borel sets again. Note that Borel sets are usually not preserved under Borel maps.

Lemma 3.10. *Under the assumptions of Lemma 3.9, if $G_0 \subseteq \mathbb{S}(X)$ is an $\overline{\lfloor\gamma\rfloor}$-invariant Borel set, then $\mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0]$ is a Borel set in $\mathbb{S}(X/\lfloor\gamma\rfloor)$.*

Proof. (0) Since G_0 is a Borel set, we know that $\mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0]$ is an analytic set in $\mathbb{S}(X/\lfloor\gamma\rfloor)$. We show that the complement of this set is analytic as well. If we would know that $\mathbb{S}(\eta_{\lfloor\gamma\rfloor})[\mathbb{S}(X)\backslash G_0]$ is disjoint from $\mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0]$, we would be done by Souslin's Theorem 2.2, so the crucial and non-obvious property is disjointness. We actually establish a bit more by showing

$$\mathbb{S}(\eta_{\lfloor\gamma\rfloor})[\mathbb{S}(X) \setminus G_0] = \mathbb{S}(X/\lfloor\gamma\rfloor) \setminus \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0].$$

(1) Because $\mathbb{S}(\eta_{\lfloor\gamma\rfloor}) : \mathbb{S}(X) \to \mathbb{S}(X/\lfloor\gamma\rfloor)$ is onto, we obtain

$$\begin{aligned}\mathbb{S}(X/\lfloor\gamma\rfloor) \setminus \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0] &= \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[\mathbb{S}(X)] \setminus \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0] \\ &\subseteq \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[\mathbb{S}(X) \setminus G_0].\end{aligned}$$

(2) Now assume that the converse inclusion is false, hence we can find $\tau \in \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[\mathbb{S}(X) \setminus G_0]$ such that $\tau \in \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0]$ also holds. By the first membership there exists $\nu \notin G_0$ with $\tau = \mathbb{S}(\eta_{\lfloor\gamma\rfloor})(\nu)$, by the second property there exists $\kappa \in G_0$ with $\tau = \mathbb{S}(\eta_{\lfloor\gamma\rfloor})(\kappa)$. Thus we find $\mu(C) = \mathbb{S}(\eta_{\lfloor\gamma\rfloor})(\nu)(C) = \nu(\eta_{\lfloor\gamma\rfloor}^{-1}[C])$ for all $C \in \mathcal{B}(X/\lfloor\gamma\rfloor)$, and similarly $\mu(C) = \mathbb{S}(\eta_{\lfloor\gamma\rfloor})(\kappa)(C) = \kappa(\eta_{\lfloor\gamma\rfloor}^{-1}[C])$ for all these C. We know that

$$\mathcal{INV}(\mathcal{B}(X), \lfloor\gamma\rfloor) = \{\eta_{\lfloor\gamma\rfloor}^{-1}[C] \mid C \in \mathcal{B}(X/\lfloor\gamma\rfloor)\},$$

hence we have found that $\kappa(D) = \nu(D)$ for all $D \in \mathcal{INV}(\mathcal{B}(X), \lfloor\gamma\rfloor)$. This implies $\nu\ \overline{\lfloor\gamma\rfloor}\ \kappa$. Since $\kappa \in G_0$, and G_0 is γ-invariant, we conclude $\nu \in G_0$, arriving at a contradiction.

(3) Hence we have established that

$$\mathbb{S}(X/\lfloor\gamma\rfloor) \setminus \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0] = \mathbb{S}(\eta_{\lfloor\gamma\rfloor})[\mathbb{S}(X) \setminus G_0],$$

so that $\mathbb{S}(\eta_{\lfloor\gamma\rfloor})[G_0]$ is the complement of an analytic set. An application of the Souslin Theorem 2.2 then yields that the set under consideration is a Borel set. □

Proposition 3.11. *Let X be an analytic space, and γ a grounded equivalence relation on $\mathbb{S}(X)$. Then $\partial_\gamma : \mathbb{S}(X)/\gamma \to \mathbb{S}(X/\lfloor\gamma\rfloor)$ is a Borel isomorphism.*

Proof. We know from Lemma 3.8 that ∂_γ is bijective and measurable, from Lemma 3.10 it is inferred that the image of a Borel set under ∂_γ is Borel as well, so that ∂_γ^{-1} is also a Borel map. □

We will use this characterization of the factor space $\mathbb{S}(X)/\gamma$ as essentially a space of probability measures when dealing in Section 6 with the factor map that is associated with factoring the Kleisli version of a randomized morphism through a congruence. Then this will turn out to be a Kleisli version of a factored randomized morphism, if the underlying congruence has grounded components.

3.2. Characterizing groundedness

For the investigations of the conditions under which a smooth relation on $\mathbb{S}(X)$ is grounded, we put

$$\Omega := \left\{ \langle \alpha_1, \ldots, \alpha_k \rangle \mid k \in \mathbb{N}, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i \leq 1 \right\}$$

for the rest of the paper, the elements of Ω being called *positive convex tuples* or simply *positive convex*.

Definition 3.12. An equivalence relation γ on $\mathbb{S}(X)$ is said to be *positive convex* iff $\mu_i \ \gamma \ \mu_i'$ for $1 \leq i \leq n$ and $\langle \alpha_1, \ldots, \alpha_n \rangle \in \Omega$ together imply

$$(\alpha_1 \cdot \mu_1 + \cdots + \alpha_n \cdot \mu_n) \ \gamma \ \left(\alpha_1 \cdot \mu_1' + \cdots + \alpha_n \cdot \mu_n'\right)$$

for each $n \in \mathbb{N}$. A partition of $\mathbb{S}(X)$ is called *positive convex* iff its associated equivalence relation is.

It is clear that $\gamma = \overline{\alpha}$ entails γ being positive convex. But it works the other way as well: a positive convex and smooth equivalence relation $\gamma = \mathsf{ker}(H)$ with a surjective Borel map $H : \mathbb{S}(X) \to T$ and a Polish space T induces a positive convex structure on T.

But first, positive convexity will be described abstractly, following Pumplün's approach [25].

Definition 3.13. A *positive convex structure* $\mathcal{P}$ on a set T assigns to each $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle \in \Omega$ a map $\alpha_{\mathcal{P}} : T^n \to T$ which we write as

$$\alpha_{\mathcal{P}}(t_1, \ldots, t_n) = \sum_{1 \leq i \leq n}^{\mathcal{P}} \alpha_i \cdot t_i,$$

such that

(i) $\sum_{1 \leq i \leq n}^{\mathcal{P}} \delta_{i,k} \cdot t_i = t_k$, where $\delta_{i,j}$ is Kronecker's δ (thus $\delta_{i,j} = 1$ if $i = j$, and $\delta_{i,j} = 0$, otherwise),

(ii) the identity

$$\sum_{1 \leq i \leq n}^{\mathcal{P}} \alpha_i \cdot \left(\sum_{1 \leq k \leq m}^{\mathcal{P}} \beta_{i,k} \cdot t_k \right) = \sum_{1 \leq k \leq m}^{\mathcal{P}} \left(\sum_{i=1}^{n} \alpha_i \beta_{i,k} \right) \cdot t_k$$

holds whenever $\langle \alpha_1, \ldots, \alpha_n \rangle, \langle \beta_{i,1}, \ldots, \beta_{i,m} \rangle \in \Omega$, $1 \leq i \leq n$.

Property (i) looks quite trivial, when written down this way. Rephrasing it states that the map

$$\langle \delta_{1,k}, \ldots, \delta_{n,k} \rangle_{\mathcal{P}} : T^n \to T,$$

which is assigned to the n-tuple $\langle \delta_{1,k}, \ldots, \delta_{n,k} \rangle$ through $\mathcal{P}$ acts as the projection to the kth component for $1 \leq k \leq n$. Similarly, property (ii) may be re-coded in a formal but less concise way. One usually uses the notation from vector spaces rather freely, omitting the explicit reference to the structure whenever possible. Hence simple addition $\alpha_1 \cdot x_1 + \alpha_2 \cdot x_2$ is written rather than $\sum_{1 \leq i \leq 2}^{\mathcal{P}} \alpha_i \cdot x_i$, with the understanding that it refers to a fixed positive convex structure $\mathcal{P}$ on T.

It can be shown that for a positive convex structure the usual rules for manipulating sums in vector spaces apply, e.g. $1 \cdot t = t$, $\sum_{i=1}^{n} \alpha_i \cdot t_i = \sum_{i=1, \alpha_i \neq 0}^{n} \alpha_i \cdot t_i$, or the law of associativity, $(\alpha_1 \cdot t_1 + \alpha_2 \cdot t_2) + \alpha_3 \cdot t_3 = \alpha_1 \cdot t_1 + (\alpha_2 \cdot t_2 + \alpha_3 \cdot t_3)$ [25]. Nevertheless, care should be observed, for of course not all rules apply: we cannot in general conclude $t = t'$ from $\alpha \cdot t = \alpha \cdot t'$, even if $\alpha \neq 0$.

Proposition 3.14. *Let γ be a positive convex and smooth equivalence relation on $\mathbb{S}(M)$ such that $\gamma = \mathsf{ker}(H)$ for some surjective Borel map $H : \mathbb{S}(M) \to T$ with M, T measurable. Then*

$$\sum_{1 \leq i \leq n}^{H} \alpha_i \cdot H(\mu_i) := H\left(\sum_{i=1}^{n} \alpha_i \cdot \mu_i \right)$$

($\alpha_1, \ldots, \alpha_n \in \Omega$, $\mu_1, \ldots, \mu_n \in \mathbb{S}(M)$) defines a positive convex structure on T.

Proof. (1) Because γ is positive convex, and $\gamma = \mathsf{ker}(H)$, we may infer from $H(\mu_i) = H(\mu_i')$ for $1 \le i \le n$ and $\alpha_1, \ldots, \alpha_n \in \Omega$ that $H\left(\sum_{i=1}^n \alpha_i \cdot \mu_i\right) = H\left(\sum_{i=1}^n \alpha_i \cdot \mu_i'\right)$, thus $\sum^H$ is well defined. It is also immediate that property (i) in Definition 3.13 is satisfied.

(2) Now let $\langle \alpha_1, \ldots, \alpha_n\rangle, \langle \beta_{i,1}, \ldots, \beta_{i,m}\rangle \in \Omega$, $1 \le i \le n$ and assume that $t_i = H(\mu_i)$ holds. Then

$$\begin{aligned}
\sum_{1\le i\le n}^{H} \alpha_i \cdot \left(\sum_{1\le k\le m}^{H} \beta_{i,k} \cdot t_k\right) &= \sum_{1\le i\le n}^{H} \alpha_i \cdot H\left(\sum_{k=1}^{m} \beta_{i,k} \cdot \mu_k\right) \\
&= H\left(\sum_{i=1}^{n} \alpha_i \cdot \left(\sum_{k=1}^{m} \beta_{i,k} \cdot \mu_k\right)\right) \\
&= H\left(\sum_{k=1}^{m} \left(\sum_{i=1}^{n} \alpha_i \cdot \beta_{i,k}\right) \cdot \mu_k\right) \\
&= \sum_{1\le k\le m}^{H} \left(\sum_{i=1}^{n} \alpha_i \cdot \beta_{i,k}\right) \cdot t_k. \quad \square
\end{aligned}$$

Because the positive convex structures are essentially the Eilenberg–Moore algebras for the Giry monad, this construction emphasizes the close relationship between positive convex and smooth equivalence relations on $\mathbb{S}(X)$ and these algebras, provided X is Polish [6].

Assume that $T = \mathbb{S}(Y)$ for some Polish space Y. This space carries a natural positive convex structure nat which assigns to $\langle \alpha_1, \ldots, \alpha_n\rangle \in \Omega$ the map $\langle \alpha_1, \ldots, \alpha_n\rangle_{\mathsf{nat}}$ with

$$\langle \alpha_1, \ldots, \alpha_n\rangle_{\mathsf{nat}}(\mu_1, \ldots, \mu_n) := \sum_{i=1}^{n} \alpha_i \cdot \mu_i,$$

so that $\sum_{1\le i\le n}^{\mathsf{nat}} \alpha_i \cdot \mu_i = \sum_{i=1}^n \alpha_i \cdot \mu_i$. It obviously has all the properties of a positive convex structure. It corresponds in a natural way to the free Eilenberg–Moore algebra $\langle \mathbb{S}(Y), \mathfrak{m}_Y\rangle$ with $\mathfrak{m}$ as multiplication for the Giry monad.

This yields a characterization of smooth equivalence relations on $\mathbb{S}(X)$ that are generated from their cousins on the base space in terms of point-affine maps.

Definition 3.15. Call a surjective map $H : \mathbb{S}(M) \to \mathbb{S}(T)$ for the measurable spaces M and T *point-affine* iff

(i) $\sum^H = \sum^{\mathsf{nat}}$,
(ii) for each $m \in M$ there exists $t \in T$ such that $H(\delta_m) = \delta_t$.

The condition (i) in Definition 3.15 is tantamount to saying that H is affine, i.e. distributes over positive convex combinations. This points directly to the Giry monad, as we will see now. A positive convex and smooth equivalence relation $\gamma = \mathsf{ker}(H)$ with $H : \mathbb{S}(X) \to T$ induces on T a positive convex structure by Proposition 3.14.

Proposition 3.16. *Let $\gamma = \mathsf{ker}(H)$ be a positive convex and smooth equivalence relation on $\mathbb{S}(X)$ with $H : \mathbb{S}(X) \to \mathbb{S}(Y)$ surjective and continuous for the Polish space X and the separable metric space Y. Then these conditions are equivalent*

(i) *γ is grounded.*
(ii) *H is point-affine.*

Proof. (1) The proof for '(i) ⇒ (ii)' is straightforward, thus '(ii) ⇒ (i)' needs to be taken care of.

(2) Define $f : X \to Y$ through $\delta_{f(x)} = H(\delta_x)$, then f is well defined and continuous. This is so since $x_n \to x$ in X implies $\delta_{x_n} \to \delta_x$ in the weak topology on $\mathbb{S}(X)$, so that

$$\delta_{f(x_n)} = H(\delta_{x_n}) \to H(\delta_x) = \delta_{f(x)},$$

from which continuity of f follows, because $\{\delta_y \mid y \in Y\}$ is homeomorphic to Y. This implies that $\mathbb{S}(f) : \mathbb{S}(X) \to \mathbb{S}(Y)$ is continuous as well.

We show that $H = \mathbb{S}(f)$, from which the assertion will follow through Lemma 3.4. In fact, from the definition of f we see that $\mathbb{S}(f)(\delta_x) = \delta_{f(x)} = H(\delta_x)$ holds, and we infer from property (ii) that for $(\alpha_1, \ldots, \alpha_n) \in \Omega$

$$\mathbb{S}(f)\left(\sum_{i=1}^{n} \alpha_i \cdot \delta_{x_i}\right) = \sum_{i=1}^{n} \alpha_i \cdot \delta_{f(x_i)} = \sum_{1 \le i \le n}^{\mathsf{nat}} \alpha_i \cdot H(\delta_{x_i})$$
$$= \sum_{1 \le i \le n}^{H} \alpha_i \cdot H(\delta_{x_i}) = H\left(\sum_{i=1}^{n} \alpha_i \cdot \delta_{x_i}\right).$$

This implies that $\mathbb{S}(f)(\mu) = H(\mu)$ holds for discrete measures μ. Since each measure can be approximated by a discrete one in the weak topology [24, Theorem II.6.3], and since both H and $\mathbb{S}(f)$ are continuous, the assertion follows. $\square$

The proof of Proposition 3.16 makes essential use of the weak topology on $\mathbb{S}(X)$, the space on which the relation is defined. This seems to indicate that having installed a Polish topology on the base space X is essential, but this is not the case: we are going to generalize this result to analytic spaces now, using a technique to move a smooth relation from a Polish space to an analytic one, developed when investigating simple relations [8, Lemma 5.4].

Lemma 3.17. *Let $f : X \to A$ and $h : A \to W$ be surjective Borel maps for the analytic spaces A, X, W. Then $\mathcal{INV}(\mathcal{B}(X), \mathsf{ker}(h \circ f)) = f^{-1}\left[\mathcal{INV}(\mathcal{B}(A), \mathsf{ker}(h))\right]$.* $\square$

Using this technique of moving relations between X and A, and similarly, between $\mathbb{S}(X)$ and $\mathbb{S}(A)$ will help in characterizing grounded equivalence relations on the analytic space A. Note that moving an equivalence relation reverses the direction for the map characterizing an analytic space.

Lemma 3.18. *Let A, X and $f : X \to A$ be as in Lemma 3.17. If γ is an equivalence relation on $\mathbb{S}(A)$, define the equivalence relation $\gamma_{\mathbb{S}(f)}$ on $\mathbb{S}(X)$ through*

$$\mu \; \gamma_{\mathbb{S}(f)} \; \mu' \Leftrightarrow \mathbb{S}(f)(\mu) \; \gamma \; \mathbb{S}(f)(\mu')$$

($\mu, \mu' \in \mathbb{S}(X)$). Then the following holds

(i) *If γ is smooth, so is $\gamma_{\mathbb{S}(f)}$.*
(ii) $\lfloor \gamma_{\mathbb{S}(f)} \rfloor = \lfloor \gamma \rfloor_f$.
(iii) *γ is near-grounded iff $\gamma_{\mathbb{S}(f)}$ is near-grounded.*
(iv) *γ is grounded iff $\gamma_{\mathbb{S}(f)}$ is grounded.*

Proof. (1) The first part is trivial, because γ is assumed to be smooth. The second part follows immediately from the observation that $\mathbb{S}(f)(\delta_x) = \delta_{f(x)}$ and the definition of $\lfloor \gamma \rfloor_f$.

(2) Assume that γ is near-grounded, and take $\mu, \mu' \in \mathbb{S}(X)$. Then

$$\mu \; \gamma_{\mathbb{S}(f)} \; \mu' \Leftrightarrow \mathbb{S}(f)(\mu) \; \gamma \; \mathbb{S}(f)(\mu') \tag{1}$$
$$\Rightarrow \forall D \in \mathcal{INV}(\mathcal{B}(A), \lfloor \gamma \rfloor) : \mu(f^{-1}[D]) = \mu'(f^{-1}[D]) \tag{2}$$
$$\Leftrightarrow \forall E \in f^{-1}[\mathcal{INV}(\mathcal{B}(A), \lfloor \gamma \rfloor)] : \mu(E) = \mu'(E) \tag{3}$$
$$\Leftrightarrow \forall E \in \mathcal{INV}(\mathcal{B}(A), \lfloor \gamma \rfloor_f) : \mu(E) = \mu'(E) \tag{4}$$
$$\Leftrightarrow \forall E \in \mathcal{INV}(\mathcal{B}(A), \lfloor \gamma_{\mathbb{S}(f)} \rfloor) : \mu(E) = \mu'(E) \tag{5}$$
$$\Leftrightarrow \mu \; \overline{\lfloor \gamma_{\mathbb{S}(f)} \rfloor} \; \mu'. \tag{6}$$

Implication (2) makes use of the assumption that γ is near-grounded, i.e. that $\gamma \subseteq \overline{\lfloor \gamma \rfloor}$ holds, equivalence (3) shifts the inverse operator f^{-1} from sets to the σ-algebra, and the next equivalence (4) applies the characterization of invariant sets from Lemma 3.17. Thus we see that the near-groundedness of γ implies the near-groundedness of $\gamma_{\mathbb{S}(f)}$.

In a similar way the converse statement is proved. Let $\mu, \mu' \in \mathbb{S}(A)$, then we can find $\mu_0, \mu'_0 \in \mathbb{S}(X)$ such that $\mu = \mathbb{S}(f)(\mu_0)$, $\mu' = \mathbb{S}(f)(\mu'_0)$. Now assume that $\gamma_{\mathbb{S}(f)}$ is near-grounded, then

$$\begin{aligned}
\mu\ \gamma\ \mu' &\Leftrightarrow \mu_0\ \gamma_{\mathbb{S}(f)}\ \mu'_0 && (7)\\
&\Rightarrow \mu_0\ \overline{\lfloor\gamma\rfloor_f}\ \mu'_0 && (8)\\
&\Leftrightarrow \forall E \in f^{-1}\left[\mathcal{INV}(\mathcal{B}(A), \lfloor\gamma\rfloor)\right] : \mu_0(E) = \mu'_0(E) && (9)\\
&\Leftrightarrow \forall G \in \mathcal{INV}(\mathcal{B}(A), \lfloor\gamma\rfloor) : \mu(G) = \mu'(G) && (10)\\
&\Leftrightarrow \forall G \in \mathcal{INV}(\mathcal{B}(A), \lfloor\gamma\rfloor) : \mathbb{S}(f)(\mu_0)(G) = \mathbb{S}(f)(\mu'_0)(G). && (11)
\end{aligned}$$

Implication (8) uses the inclusion $\gamma_{\mathbb{S}(f)} \subseteq \overline{\lfloor\gamma_{\mathbb{S}(f)}\rfloor} = \overline{\lfloor\gamma\rfloor_f}$, and equivalence (9) applies the characterization of invariant sets from Lemma 3.17. The other arguments are straightforward. Thus the near-groundedness of $\gamma_{\mathbb{S}(f)}$ implies the near-groundedness of γ. This establishes part (iii).

(3) For the proof of part (iv), it is noted that for grounded relations the implications (2) resp. (8) can be reversed. □

Using this characterization of a relation induced by the inverse image of a Borel map we can generalize Proposition 3.16 to analytic base spaces.

Proposition 3.19. *Let $\gamma = \mathsf{ker}(H)$ be a positive convex and smooth equivalence relation on $\mathbb{S}(A)$ with $H : \mathbb{S}(A) \to \mathbb{S}(Y)$ surjective and Borel for the analytic space A and the analytic space Y. Then γ is grounded iff H is point-affine.*

Proof. (1) Let first Y be a separable metric space. Assume that $\gamma = \mathsf{ker}(H)$ is grounded. Since A is an analytic space, we can find a surjective Borel map $f : X \to A$ that defines the measurable structure on A. Because of Lemma 2.1 we may assume that $H \circ \mathbb{S}(f) \circ \mathfrak{e}_X$ is continuous. Thus the assertion follows from the observation that $\sum^{H\circ\mathbb{S}(f)} = \sum^{H}$ together with Lemma 3.18, part (iv) and the characterization of grounded relations in Proposition 3.16.

(2) If Y is an analytic space, then we can find a separable metric space Z such that $(Y, \mathcal{B}(Y))$ is Borel-isomorphic to $(Z, \mathcal{B}(Z))$ by [15, Proposition 12.1]. Thus $\mathbb{S}(Y)$ and $\mathbb{S}(Z)$ are Borel isomorphic, and the assertion follows from part (1). □

4. Randomized congruences and morphisms

All preparations are now in place for a definition of randomized congruences and randomized morphisms based on the Kleisli category for the Giry monad. In order to define congruences, we will first extend the stochastic relation $K : X \rightsquigarrow Y$, i.e. the map $K : X \to \mathbb{S}(Y)$ to a map $\overline{K} : \mathbb{S}(X) \to \mathbb{S}(Y)$ (its Kleisli extension) in the canonical way. In fact, [20,14] use this extension as a starting point. The extension renders a map for which a congruence can be defined in a way that is customary in Universal Algebra [10]. We will, however, want to take care that we interlock the base space and the space of all subprobabilities on it properly, for otherwise the definition of a congruence would be too general to be of any use here. We will define randomized morphisms as those stochastic relations for which composition yields a commutative diagram, the composition being given through the Kleisli construction. In the same spirit there is some care to be exercised in not permitting too large a gap to open between the base space and its subprobabilities. It turns out that both constructions are really the randomizations of the constructions that are well known in their non-randomized version (so that, roughly, considering Dirac measures δ_x in the randomized world corresponds very closely to considering the elements proper in the non-randomized context). Moreover, each randomized morphism spawns a randomized congruence in a natural way.

Define the *Kleisli extension* $\overline{K}$ of a stochastic relation $K : X \rightsquigarrow Y$ through

$$\overline{K} := \mathfrak{m}_Y \circ \mathbb{S}(K)$$

with $\mathfrak{m}_Y : \mathbb{S}^2(Y) \to \mathbb{S}(Y)$ as the multiplication in the Giry monad. Let $L : Y \rightsquigarrow Z$ be another stochastic relation, then [19, Theorem VI.5.1] the Kleisli product $L * K$ is defined through

$$L * K := \mathfrak{m}_Z \circ \mathbb{S}(L) \circ K.$$

The definition given above in Section 2 is easily derived from it using properties of the multiplication $\mathfrak{m}_Z$. We observe the following properties [9], which in particular relate the composition of a Kleisli extension with the Kleisli extension of the Kleisli product.

Lemma 4.1. *Let $K : X \rightsquigarrow Y$ and $L : Y \rightsquigarrow Z$ be stochastic relations.*

(i) $\overline{K}(\mu)(B) = \int_X K(x)(B)\, \mu(\mathrm{d}x)$*, whenever $B \subseteq Y$ is a measurable set.*
(ii) $\overline{L} \circ \overline{K} = \overline{L * K}$.
(iii) *If X and Y are Polish, and K is continuous, so is $\overline{K}$.*

Proof. (1) Let $\mu \in \mathbb{S}(X)$ a subprobability over X, $B \in \mathcal{B}(Y)$ be a Borel set, then

$$\begin{aligned}\overline{K}(\mu)(B) &= \int_{\mathbb{S}(X)} \tau(B)\, \mathbb{S}(K)(\mu)(\mathrm{d}\tau)\\ &= \int_X K(x)(B)\, \mu(\mathrm{d}x).\end{aligned}$$

The first equation follows from the definition of the multiplication in the Giry monad, the second one comes from the Change of Variables theorem. This establishes property (i). Property (ii) is proved by a direct computation, taking into account that

$$\int_Y f\, \mathrm{d}\overline{K}(\mu) = \int_X \int_Y f(y)\, K(x)(\mathrm{d}y)\, \mu(\mathrm{d}x),$$

whenever $f : Y \to \mathbb{R}$ is bounded and measurable.

(2) If $K : X \to \mathbb{S}(Y)$ is continuous, so is $\mathbb{S}(K) : \mathbb{S}(X) \to \mathbb{S}^2(Y)$. Thus assertion (iii) follows from the observation that $\mathfrak{m}_Y$ is continuous in the weak topology, whenever Y is Polish [8, Lemma 3.2]. $\square$

An alternative proof of part (ii) would take only properties of the monad's multiplication into account [13].

4.1. Randomized morphisms

We will define randomized morphisms as those morphisms that correspond to the Kleisli category for the Giry monad through stochastic relations. Let $\mathsf{K} = (X, Y, K)$ and $\mathsf{L} = (A, B, L)$ be stochastic relations, and assume that $\Phi : X \rightsquigarrow A$ and $\Psi : Y \rightsquigarrow B$. Then we can compose K, Φ, L, Ψ in different ways, the most natural one being through the Kleisli product, where we stipulate the commutativity of the corresponding diagrams.

This leads to the definition of a randomized morphism between stochastic relations.

Definition 4.2. Assume that $\mathsf{K} = (X, Y, K)$ and $\mathsf{L} = (A, B, L)$ are stochastic relations, $\mathsf{F} = (\Phi, \Psi)$ is called a *randomized morphism* between K and L (in symbols: $\mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$) iff

(i) $\Phi : X \rightsquigarrow A$ and $\Psi : Y \rightsquigarrow B$ are stochastic relations,
(ii) $\overline{\Phi}$ and $\overline{\Psi}$ are onto,
(iii) $\Psi * K = L * \Phi$ holds.

Consequently, a randomized morphism makes the diagram on the left- hand side below commutative products taken in the Kleisli category associated with the Giry monad; composing morphisms through the Kleisli product here is indicated through dashed arrows. For comparison we display also the diagram for $\mathsf{f} = (\varphi, \psi) : \mathsf{K} \to \mathsf{L}$ to be a non-randomized morphism. Thus $\varphi : X \to A$ and $\psi : Y \to B$ are surjective Borel maps, and the diagram is commutative using ordinary map composition.

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, dashed, "K"'] & A \arrow[d, dashed, "L"] \\ Y \arrow[r, dashed, "\Psi"'] & B \end{tikzcd} \qquad \text{vs.} \qquad \begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "K"'] & A \arrow[d, "L"] \\ \mathbb{S}(Y) \arrow[r, "\mathbb{S}(\psi)"'] & \mathbb{S}(B) \end{tikzcd}$$

(randomized) (non-randomized)

Spelling out the condition for a randomized morphism in terms of integrals, we see that

$$(\Psi * K)(x)(D) = \int_Y \Psi(y)(D)\ K(x)(\mathrm{d}y) = \int_A L(a)(D)\ \Phi(x)(\mathrm{d}a) = (L * \Phi)(x)(D)$$

should hold for $x \in X$ and the Borel set $D \in \mathcal{B}(B)$. Thus averaging $\Psi(\cdot)(D)$ using $K(x)$ is tantamount to averaging $L(\cdot)(D)$ with respect to $\Phi(x)$.

Expanding the diagram on the right-hand side for the non-randomized version means that

$$L(\varphi(x))(D) = (L \circ \varphi)(x)(D) = (\mathbb{S}(\psi) \circ K)(x)(D) = K(x)(\psi^{-1}[D]),$$

thus the probability of hitting an element of D after input $\varphi(x)$ in L equals the probability of hitting an element of the inverse image $f^{-1}[D]$ after input $x \in X$. Comparing both definitions reveals that the randomized version entertains an additional level of averaging.

Surjectivity of the maps underlying f is introduced in order to make sure that each element in the range originates from some element in the domain. Similarly, surjectivity of the Kleisli versions of the stochastic relations underlying a randomized morphism takes care that each distribution on the range can be accounted for.

Randomized morphisms are related to their non-randomized cousins in a similar way that randomized policies are related to policies in stochastic dynamic programming: whereas policies assign each state an action to take, a randomized policy assigns each state a probability distribution over actions [12].

Returning to the general discussion, let $\mathsf{F} = (\Phi, \Psi) : \mathsf{K} \curvearrowright \mathsf{L}$ and $\mathsf{G} = (\Gamma, \Theta) : \mathsf{L} \curvearrowright \mathsf{M}$ be randomized morphisms, then their composition $\mathsf{G} * \mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$ is defined in the obvious way through $\mathsf{G} * \mathsf{F} := (\Gamma * \Phi, \Theta * \Psi)$.

Proposition 4.3. *Stochastic relations over general measurable spaces as objects and with randomized morphisms as morphisms form a category, when the composition of morphisms is defined through the Kleisli construction. The same is true for stochastic relations over Polish or over analytic spaces.*

Proof. The crucial property is the surjectivity of the induced maps. This property, however, follows immediately from Lemma 4.1, property (ii). All other properties are well known [9,23,8]. □

Randomized morphisms have non-randomized morphisms as special cases. It is instructive to look at the proof in some detail.

Lemma 4.4. *Let $\mathsf{K} = (X, Y, K)$, $\mathsf{L} = (A, B, L)$ be stochastic relations, and assume that $\mathsf{f} = (\varphi, \psi) : \mathsf{K} \to \mathsf{L}$ is a non-randomized morphism. Put $\Phi := \mathfrak{e}_A \circ \varphi$, $\Psi := \mathfrak{e}_B \circ \psi$, then $\mathsf{F}_\mathsf{f} : (\Phi, \Psi) : \mathsf{K} \curvearrowright \mathsf{L}$ is a randomized morphism.*

Proof. (1) Note that $\Phi(x) = \delta_{\varphi(x)}$ and $\Psi(y) = \delta_{\psi(y)}$, hence an easy calculation of the Kleisli extension demonstrates that $\overline{\Phi} = \mathbb{S}(\varphi)$ and $\overline{\Psi} = \mathbb{S}(\psi)$. Thus both $\overline{\Phi}$ and $\overline{\Psi}$ are onto on account of φ and ψ being onto.

(2) Let $\mu \in \mathbb{S}(X)$ and $G \in \mathcal{B}(B)$, then

$$\begin{aligned}
\overline{\Psi}(\overline{K})(\mu)(G) &= \int_Y \Psi(y)(G)\ \overline{K}(\mu)(\mathrm{d}y) \\
&= \overline{K}(\mu)(\psi^{-1}[G]) \\
&= \int_X K(x)(\psi^{-1}[G])\ \mu(\mathrm{d}x) \\
&= \int_X \mathbb{S}(\psi)(K(x))(G)\ \mu(\mathrm{d}x) \\
&= \int_X L(\varphi(x))(G)\ \mu(\mathrm{d}x).
\end{aligned}$$

The last equality follows from $L \circ \varphi = \mathbb{S}(\psi) \circ K$. Similarly,

$$\overline{L}(\overline{\Phi}(\mu))(G) = \int_A L(a)(G)\ \overline{\Phi}(\mu)(\mathrm{d}a)$$

$$= \int_A L(a)(G)\, \mathbb{S}(\varphi)(\mu)(\mathrm{d}a)$$
$$= \int_X L(\varphi(x))(G)\, \mu(\mathrm{d}x),$$

the last equation following from the Change of Variables formula. Thus we see that $\overline{\Psi} \circ \overline{K} = \overline{L} \circ \overline{\Phi}$ holds, from which the desired equality $\Psi * K = L * \Phi$ follows: from Lemma 4.1 $\overline{\Psi * K} = \overline{L * \Phi}$ is obtained. Thus

$$(\Psi * K)(x) = \overline{\Psi * K}(\delta_x) = \overline{L * \Phi}(\delta_x) = (L * \Phi)(x)$$

is inferred for each $x \in X$. □

This statement may be proven alternatively through properties of a monad. Integration, however, provides more insight into the probabilistic content of $\mathsf{f} \mapsto \mathsf{F}_\mathsf{f}$.

Corollary 4.5. *Map each stochastic relation over analytic spaces to itself, and map a non-randomized morphism* $\mathsf{f} : \mathsf{K} \to \mathsf{L}$ *to* $\mathsf{F}_\mathsf{f} : \mathsf{K} \curvearrowright \mathsf{L}$. *Then this constitutes a functor* $\mathfrak{F}$ *from the category of stochastic relations with non-randomized morphisms over analytic spaces to the category of stochastic relations with randomized morphisms over analytic spaces.*

Proof. This is essentially due to a favorable property of the unit in a monad. Let e.g. X, Y, Z be the input spaces for three relations, and $\varphi : X \to Y$, $\psi : Y \to Z$ be Borel maps, then

$$\mathfrak{e}_Z \circ (\psi \circ \varphi) = (\mathfrak{e}_Z \circ \psi) * (\mathfrak{e}_Y \circ \varphi).$$

From this,

$$\mathfrak{F}(\mathsf{g} \circ \mathsf{f}) = \mathfrak{F}(\mathsf{g}) * \mathfrak{F}(\mathsf{f})$$

is derived. All other properties of a functor are trivial. □

The functor $\mathfrak{F}$ may be described as a slightly confused fellow: it forgets the deterministic nature of a non-randomized morphism and interprets deterministic behavior as randomized, albeit with a somewhat limited set of possibilities from which to choose.

4.2. Randomized congruences

Morphisms and congruences are closely related. Recall from Section 2 that the pair (α, β) of smooth equivalence relations on X resp. Y is a *congruence* for $\mathsf{K} = (X, Y, K)$ iff $K(x)(B) = K(x')(B)$ holds whenever $x\ \alpha\ x'$, and $B \in \mathcal{INV}(\mathcal{B}(Y), \beta)$. In contrast, a randomized congruence for K is based on a pair of smooth equivalence relations (ρ, τ) on $\mathbb{S}(X)$ resp. $\mathbb{S}(Y)$. We require both equivalences to be near-grounded, so that we can exercise some control from the base space over these relations. In particular we are able to relate the relation to the invariant subsets on the base space, which is not only intuitively appealing but provides the desired bracket between these spaces.

Before defining a randomized congruence, let us see what happens when we lift a congruence for a stochastic relation.

Lemma 4.6. *Let* $\mathsf{K} = (X, Y, K)$ *be an analytic stochastic relation, and assume that* $\mathsf{c} := (\alpha, \beta)$ *is a pair of smooth equivalence relations on* X *resp.* Y. *Put* $\overline{\mathsf{c}} := (\overline{\alpha}, \overline{\beta})$. *These conditions are equivalent for* K*:*

(i) c *is a congruence.*
(ii) $\overline{\mathsf{c}}$ *has these properties*
  (a) $\overline{K}(\mu)\ \overline{\beta}\ \overline{K}(\mu')$ *whenever* $\mu\ \overline{\alpha}\ \mu'$,
  (b) $\overline{\alpha}$ *and* $\overline{\beta}$ *are near-grounded.*

Proof. (1) '(i) ⇒ (ii)': The map $x \mapsto K(x)(B)$ is $\mathcal{INV}(\mathcal{B}(X), \alpha)$-$\mathcal{B}(\mathbb{R})$-measurable, whenever $B \in \mathcal{INV}(\mathcal{B}(Y), \beta)$ is a β-invariant Borel set of Y by Lemma 2.8. Taking this into account, an application of Lemma 2.6 now yields for the Kleisli extensions

$$\overline{K}(\mu)(B) = \int_X K(x)(B)\, \mu(\mathrm{d}x) = \int_X K(x)(B)\, \mu'(\mathrm{d}x) = \overline{K}(\mu')(B),$$

whenever $\mu\ \overline{\alpha}\ \mu'$, and $B \in \mathcal{INV}(\mathcal{B}(Y), \beta)$. Thus $\overline{K}(\mu)\ \overline{\beta}\ \overline{K}(\mu)$. It is apparent from Corollary 3.2 that $\overline{\alpha}$ and $\overline{\beta}$ are near-grounded.

(2) '(ii) $\Rightarrow$ (i)': Observe $\overline{K}(\delta_x)\ \overline{\beta}\ \overline{K}(\delta_{x'})$, provided $x\ \alpha\ x'$, because the latter implies $\delta_x\ \overline{\alpha}\ \delta_{x'}$ by Corollary 3.2. But then

$$K(x)(B) = \overline{K}(\delta_x)(B) = \overline{K}(\delta_{x'})(B) = K(x')(B)$$

whenever $B \in \mathcal{INV}(\mathcal{B}(Y), \beta)$ is a β-invariant Borel set in Y. Thus (α, β) is a congruence for K. □

This suggests the following definition of a randomized congruence.

Definition 4.7. Let $\mathsf{C} := (\rho, \tau)$ be a pair of smooth equivalence relations on $\mathbb{S}(X)$ resp. $\mathbb{S}(Y)$ with X and Y analytic. C is said to be a *randomized congruence* for the stochastic relation $\mathsf{K} = (X, Y, K)$ iff

(i) $\mu\ \rho\ \mu'$ implies $\overline{K}(\mu)\ \tau\ \overline{K}(\mu')$ for all $\mu, \mu' \in \mathbb{S}(X)$,
(ii) both ρ and τ are near-grounded.

Congruence C is said to be *grounded* iff both ρ and τ are grounded.

Apart from near-groundedness, a congruence has the usual properties that are required for a relation that is supposed to reflect the structure in an algebraic setting [10, Section 1.7]. A stochastic relation can be considered as a coalgebra for functor $\mathbb{S}$, hence a coalgebraic view of congruences is interesting. Rutten [26] argues that congruences correspond in universal coalgebra to the largest bisimulation relation. This point of view is vital in the discussion of coalgebras for the powerset functor. It needs to be adapted to the situation at hand, given that bisimulations for stochastic relations have a somewhat more complicated structure than one would expect, and given that $\mathbb{S}$ does not enjoy some properties (such as not dealing decently with weak pullbacks) that are usually taken for granted when investigating coalgebras [7].

We see from Lemma 4.6 that lifting a congruence to the space of subprobabilities yields a randomized congruence, as one would reasonably hope. A converse property is available as well: take a randomized congruence, then its trace will define a congruence.

Corollary 4.8. *If* $\mathsf{C} = (\rho, \tau)$ *is a randomized congruence for the analytic stochastic relation* $\mathsf{K} = (X, Y, K)$*. Then* $\lfloor \mathsf{C} \rfloor := (\lfloor \rho \rfloor, \lfloor \tau \rfloor)$ *is a congruence for* K.

Proof. Let $x, x' \in X$ with $x\ \lfloor \rho \rfloor\ x'$, thus with $\delta_x\ \rho\ \delta_{x'}$. Consequently,

$$(K(x) =)\ \overline{K}(\delta_x)\ \tau\ \overline{K}(\delta_{x'}) (= K(x')),$$

thus, because τ is near-grounded, $K(x)\ \overline{\lfloor \tau \rfloor}\ K(x')$. This means that $K(x)(B) = K(x')(B)$ holds, whenever $B \in \mathcal{INV}(\mathcal{B}(Y), \lfloor \tau \rfloor)$. □

If $\mathsf{F} := (\Phi, \Psi)$ is a randomized morphism $\mathsf{K} \curvearrowright \mathsf{L}$, we define as usual its kernel $\mathsf{ker}(\mathsf{F})$ through

$$\mathsf{ker}(\mathsf{F}) := (\mathsf{ker}(\Phi), \mathsf{ker}(\Psi)).$$

Similarly, the kernel

$$\mathsf{ker}(\overline{\mathsf{F}}) := (\mathsf{ker}(\overline{\Phi}), \mathsf{ker}(\overline{\Psi}))$$

is defined for the Kleisli version. It is clear that both kernels are based on smooth equivalence relations, since Φ and Ψ as well as $\overline{\Phi}$ and $\overline{\Psi}$ are Borel maps between Polish spaces.

Randomized congruences provide the paragon for defining near-grounded morphisms.

Definition 4.9. The randomized morphism $\mathsf{F} = (\Phi, \Psi) : \mathsf{K} \curvearrowright \mathsf{L}$ is called *near-grounded* iff both $\mathsf{ker}(\overline{\Phi}) \subseteq \overline{\mathsf{ker}(\Phi)}$ and $\mathsf{ker}(\overline{\Psi}) \subseteq \overline{\mathsf{ker}(\Psi)}$.

Thus a near-grounded morphism $(\Phi, \Psi) : (X, Y, K) \curvearrowright (A, B, L)$ has in addition to $\Psi * K = L * \Phi$ the property that

$$\overline{\Phi}(\mu) = \overline{\Phi}(\mu') \Rightarrow \mu(D) = \mu'(D) \quad \text{for all } D \in \mathcal{INV}(\mathcal{B}(X), \mathsf{ker}(\Phi)),$$
$$\overline{\Psi}(\nu) = \overline{\Psi}(\nu') \Rightarrow \nu(C) = \nu'(C) \quad \text{for all } C \in \mathcal{INV}(\mathcal{B}(Y), \mathsf{ker}(\Psi)).$$

Consider $\overline{\Phi}$; the behavior of μ and μ' on those Borel sets the elements of which cannot be separated by $\mathsf{ker}(\Phi)$ is identical, provided their image under the Kleisli extension of Φ is identical (formally, $\mathbb{S}(\eta_{\mathsf{ker}(\Phi)})(\mu) = \mathbb{S}(\eta_{\mathsf{ker}(\Phi)})(\mu')$ holds whenever $\overline{\Phi}(\mu) = \overline{\Phi}(\mu')$). Thus if $\overline{\Phi}$ cannot distinguish between two distributions, these distributions do not differentiate sets that are comprised of building blocks on which Φ is constant. Similarly for Ψ. Consequently, a near-grounded randomized morphism takes the behavior of the elements of the base set with respect to its components into account in a very specific way.

Lemma 4.10. *Let* $\mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$ *be a near-grounded randomized morphism with* K *analytic, then* $\mathsf{ker}(\overline{\mathsf{F}})$ *is a randomized congruence for* K.

Proof. Assume $\overline{\Phi}(\mu) = \overline{\Phi}(\mu')$, then Lemma 4.1 yields $\overline{\Psi}(\overline{K}(\mu)) = \overline{\Psi}(\overline{K}(\mu'))$. Because F is near-grounded, both $\mathsf{ker}(\overline{\Phi})$ and $\mathsf{ker}(\overline{\Psi})$ are near-grounded. This is so since $\lfloor \mathsf{ker}(\overline{\Phi}) \rfloor = \mathsf{ker}(\Phi)$ holds, for

$$x \lfloor \mathsf{ker}(\overline{\Phi}) \rfloor x' \Leftrightarrow \overline{\Phi}(\delta_x) = \overline{\Phi}(\delta_{x'}) \Leftrightarrow \Phi(x) = \Phi(x') \Leftrightarrow x \,\mathsf{ker}(\Phi)\, x'.$$

Thus $\mathcal{INV}(\mathcal{B}(X), \mathsf{ker}(\Phi)) = \mathcal{INV}(\mathcal{B}(X), \lfloor \mathsf{ker}(\overline{\Phi}) \rfloor)$. Since F is near-grounded, we know $\mathsf{ker}(\overline{\Phi}) \subseteq \overline{\mathsf{ker}(\Phi)}$, consequently we have

$$\begin{aligned}\overline{\Phi}(\mu) = \overline{\Phi}(\mu') &\Rightarrow \forall B \in \mathcal{INV}(\mathcal{B}(X), \lfloor \mathsf{ker}(\overline{\Phi}) \rfloor) : \mu(B) = \mu'(B)\\ &\Leftrightarrow \mu \lfloor \mathsf{ker}(\overline{\Phi}) \rfloor \mu'.\end{aligned}$$

This yields $\mathsf{ker}(\overline{\Phi}) \subseteq \overline{\lfloor \mathsf{ker}(\overline{\Phi}) \rfloor}$, similarly for Ψ. Thus both $\mathsf{ker}(\overline{\Phi})$ and $\mathsf{ker}(\overline{\Psi})$ are near-grounded. $\square$

This implies that the kernel of a randomized morphism for a stochastic relation is a congruence for that relation as well. Notice the difference: while being a randomized congruence for K addresses the map $\overline{K}$, being a congruence for K entails that testing K on invariant Borel sets is necessary.

We did postulate near-groundedness above as a sensible condition to interlock the behavior on X and on $\mathbb{S}(X)$ in a suitable manner. The question arises whether groundedness would also be a suitable condition. It turns out that this condition is way too strong, since it only reflects the well-known phenomenon of non-randomized morphisms, suitably dressed up.

Lemma 4.11. *Let for the Polish spaces* X *and* Y *be* $\Phi : X \rightsquigarrow Y$ *be a stochastic relation such that* $\overline{\Phi}$ *is onto. Then* $\mathsf{ker}(\overline{\Phi})$ *is grounded iff there exists a surjective Borel map* $f : X \to Y$ *with* $\Phi = \mathfrak{e}_Y \circ f$.

Proof. (0) Because of Lemma 2.1 we may and do assume that Φ resp. f are continuous.

(1) Proposition 3.16 implies that $\mathfrak{e}_Y \circ f$ defines a grounded equivalence relation on $\mathbb{S}(X)$, whenever f is continuous and onto.

(2) Now let Φ be continuous with surjective Kleisli extension $\overline{\Phi}$, then the argumentation in the proof of Proposition 3.16 shows that $\Phi(x) = \overline{\Phi}(\delta_x) =: \delta_{f(x)}$ defines a continuous map $f : X \to Y$ such that $\overline{\Phi}(\mu) = \mathbb{S}(f)(\mu)$. Hence surjectivity of f remains to be shown. $\mathbb{S}(f)$ is surjective, so given $y \in Y$ there exists $\mu \in \mathbb{S}(X)$ with $\mathbb{S}(f)(\mu) = \delta_y$, hence we have $\delta_y(B) = \mu(f^{-1}[B])$ for each Borel set $B \in \mathcal{B}(Y)$. In particular, $\mu(f^{-1}[\{y\}]) = 1$, so that $f^{-1}[\{y\}] \neq \emptyset$, thus there exists $x \in X$ with $f(x) = y$. $\square$

Proposition 4.12. *Let* $\mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$ *be a randomized morphism, then* $\mathsf{ker}(\overline{\mathsf{F}})$ *is grounded iff there exists a morphism* $\mathsf{f} : \mathsf{K} \to \mathsf{L}$ *such that* $\mathsf{F} = \mathsf{F}_{\mathsf{f}}$, *i.e.,* F *is deterministic.*

Proof. This follows immediately from Lemma 4.11 in conjunction with Lemma 4.4. $\square$

There is, however, a useful interconnection between non-randomized congruences and kernels of near-randomized morphisms. This is quite easy to establish and will be rather helpful below.

Proposition 4.13. *Let* $\mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$ *be a near-grounded morphism, then* $\mathsf{ker}(\mathsf{F})$ *is a congruence for* K, *and* $\lfloor \mathsf{ker}(\overline{\mathsf{F}}) \rfloor = \mathsf{ker}(\mathsf{F})$.

Proof. From Lemma 4.10 we infer that $\mathsf{ker}(\overline{\mathsf{F}})$ is a randomized congruence. Let $\mathsf{F} = (\Phi, \Psi)$. A simple calculation shows that $\lfloor \mathsf{ker}(\overline{\Phi}) \rfloor = \mathsf{ker}(\Phi)$ and $\lfloor \mathsf{ker}(\overline{\Psi}) \rfloor = \mathsf{ker}(\Psi)$ both hold. $\square$

5. A simple logic

We discuss as an illustration of the concepts of randomized congruences and randomized morphisms a well-known logic and its interpretation through labeled Markov processes (which are sometimes called stochastic Kripke models, in particular when primitive propositions are taken additionally into account). This logic has been derived from a version that was originally introduced by Hennessy and Milner [11] for the study of bisimulations. It was modified by Larsen and Skou [18] in their seminal paper on testing, where a very close connection between bisimilarity and logical equivalence (accepting the same sets of formulas) for models was formulated in the context of (discrete) labeled Markov processes. This was then generalized in [4,7] to models over general analytic spaces with an eye towards bisimilarity.

At the very core of these investigations are two observations:

(1) The equivalence relation that makes states equivalent iff they satisfy exactly the same formulas is smooth.
(2) If f is a morphism between models, then a state s satisfies a formula iff $f(s)$ satisfies the same formula.

The first observation permits to decently factor the state spaces, building up a probabilistic structure on the factor space, the second observation permits defining a bisimulation in a suitable way over the sum of certain factor spaces (the reader is referred to the corresponding proofs in [4] or in [7]).

We discuss these properties in this section in a randomized setting and follow first [4] in introducing syntax and semantics of the Hennessy–Milner logic $\mathcal{L}$. We will then show that a randomized congruence is associated with the logic (which is not too much of a surprise, taking Lemma 4.6 into account), and discuss how randomized morphisms act on these congruences. This requires specializing the notion of a randomized morphism to labeled Markov processes, and to adapt a morphism to the logic.

The syntax of the Hennessy–Milner logic $\mathcal{L}$ is given by

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a\rangle_q \phi.$$

Here $a \in \mathsf{A}$ is an action, and q is a rational number; the set A of labels is assumed to be at most countably infinite. The semantics of $\langle a\rangle_q \phi$ says intuitively that we can make an a-move in a state s to a state that satisfies ϕ with probability greater than q.

A *labeled Markov transition system* $\mathtt{M} := (S, (k_a)_{a\in\mathsf{A}})$ is comprised of a Polish state space S, and for each action a there is a stochastic relation $k_a : S \rightsquigarrow S$. The interpretation of k_a reads that $k_a(s)(E)$ is the probability that upon action $a \in \mathsf{A}$ in state $s \in S$ the next state is a member of $E \in \mathcal{B}(S)$. Fix the labeled Markov process $\mathtt{M}$ with state space S and transition laws $k_a : S \rightsquigarrow S$.

Satisfaction of a state s for a formula ϕ is defined inductively, $[\![\phi]\!]_{\mathtt{M}}$ is defined as the set of all states s that satisfy formula ϕ, i.e. for which $\mathtt{M}, s \models \phi$ holds. This is trivial for $\top$ and for formulas of the form $\phi_1 \wedge \phi_2$. The more complicated case is making an a-move: $\mathtt{M}, s \models \langle a\rangle_q \phi$ holds iff $k_a(s)([\![\phi]\!]_{\mathtt{M}}) \geq q$ holds.

The set of formulas $\mathbf{F}_{\mathcal{L}}$ defines an equivalence relation $\lambda_{\mathtt{M}}$ on the states upon

$$s\ \lambda_{\mathtt{M}}\ s' \text{ iff } \forall \phi \in \mathbf{F}_{\mathcal{L}} : \left[\mathtt{M}, s \models \phi \Leftrightarrow \mathtt{M}, s' \models \phi\right].$$

It is well known that $[\![\phi]\!]_{\mathtt{M}}$ is an $\lambda_{\mathtt{M}}$-invariant Borel set for each formula ϕ.

Relation $\lambda_{\mathtt{M}}$ is a smooth equivalence relation, since $\mathbf{F}_{\mathcal{L}}$ is a countable set, moreover, it is a congruence for each $k_a : S \rightsquigarrow S$. This statement is well known, we give a short proof for the sake of completeness.

Lemma 5.1. *Let $\mathtt{M} = (S, (k_a)_{a\in\mathsf{A}})$ be a labeled Markov transition system, then the equivalence relation $\lambda_{\mathtt{M}}$ is a k_a-congruence for each action $a \in \mathsf{A}$.*

Proof. (1) Fix action $a \in \mathsf{A}$. We show first that $x\ \lambda_{\mathtt{M}}\ x'$ implies that $k_a(x)([\![\phi]\!]_{\mathtt{M}}) = k_a(x')([\![\phi]\!]_{\mathtt{M}})$, whenever ϕ is a formula in $\mathcal{L}$. Suppose that this is not true, then there is a formula ϕ for which the equality above is false. Thus we can squeeze in a rational number q with $k_a(x)([\![\phi]\!]_{\mathtt{M}}) < q \leq k_a(x')([\![\phi]\!]_{\mathtt{M}})$ (or vice versa), so that $\mathtt{M}, x \not\models \langle a\rangle_q \phi$ but $\mathtt{M}, x' \models \langle a\rangle_q \phi$. This is a contradiction.

(2) Now consider for fixed x, x' with $x\ \lambda_{\mathtt{M}}\ x'$ the set

$$\mathcal{D} := \{D \in \mathcal{INV}(\mathcal{B}(S), \lambda_{\mathtt{M}}) \mid k_a(x)(D) = k_a(x')(D)\}.$$

Then $[\![\phi]\!]_{\mathtt{M}} \in \mathcal{D}$, for each formula ϕ by the first part, $\mathcal{D}$ is closed under complementation and countable disjoint unions, since both $k_a(x)$ and $k_a(x')$ are measures. The π-λ-Theorem shows that $\sigma(\{[\![\phi]\!]_{\mathtt{M}} \mid \phi \in \mathbf{F}_{\mathcal{L}}\}) \subseteq \mathcal{D}$. Because

$$\mathcal{INV}(\mathcal{B}(S), \lambda_{\mathtt{M}}) = \sigma(\{[\![\phi]\!]_{\mathtt{M}} \mid \phi \in \mathbf{F}_{\mathcal{L}}\}),$$

this implies the claim.

Consequently, $x\ \lambda_{\mathtt{M}}\ x'$ implies $k_a(x)(D) = k_a(x')(D)$ for all $\lambda_{\mathtt{M}}$-invariant Borel sets D of S. $\square$

Now we relate two labeled Markov processes through a randomized morphism; this is simply a randomized morphism for any relation corresponding to the same action.

Definition 5.2. Let $\mathtt{M} = (S, (k_a)_{a\in\mathsf{A}})$ and $\mathtt{N} = (T, (\ell_a)_{a\in\mathsf{A}})$ be labeled Markov transition systems. The stochastic relation $\Phi : S \rightsquigarrow T$ is called a *randomized morphism from* $\mathtt{M}$ *to* $\mathtt{N}$, in symbols $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ iff $\Phi : (S, k_a) \curvearrowright (T, \ell_a)$ for each $a \in \mathsf{A}$.

Thus given $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$, we have $\ell_a * \Phi = \Phi * k_a$ for each action $a \in \mathsf{A}$, or, equivalently, that $\overline{\ell_a} \circ \overline{\Phi} = \overline{\Phi} \circ \overline{k_a}$. Again, this is compared to the non-randomized situation. Recall that $f : \mathtt{M} \to \mathtt{N}$ is a morphism for transition systems $\mathtt{M}$ and $\mathtt{N}$ iff $f : (S, k_a) \to (T, \ell_a)$ is a morphism for each $a \in \mathsf{A}$, cp. [4,7]. Thus $f : S \to T$ is a surjective Borel map such that $\ell_a \circ f = \mathbb{S}(f) \circ k_a$ for each action a. Consequently, given action $a \in \mathsf{A}$, we know for a Borel set $B \subseteq T$ and a state $s \in S$ that $\ell_a(f(s))(B) = k_a(s)(f^{-1}[B])$ holds, so that the probability to hit a state $t \in B$ from state $f(s)$ in $\mathtt{N}$ equals the probability to hit a state s' with $f(s') \in B$ from state s in $\mathtt{N}$. For $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ we have ($s \in S$, $B \in \mathcal{B}(T)$)

$$\int_T \ell_a(t)(B)\ \Phi(s)(\mathrm{d}t) = \int_S \Phi(s')(B)\ k_a(s)(\mathrm{d}s')$$

so that the probability to hit from state s a new state that is a member of set B average over through $\Phi(s)$ equals the average probability to hit this state with the morphism, when averaged through $k_a(s)$.

This lemma is a reformulation for a well-known fact, viz, that validity of a formula is respected by morphisms.

Lemma 5.3. *Let* $\mathtt{M}$ *and* $\mathtt{N}$ *be Markov transition systems over the Polish state spaces* S *resp.* T, *and let* $f : \mathtt{M} \to \mathtt{N}$ *be a non-randomized morphism, then*

(i) $\mathtt{M}, s \models \phi \Leftrightarrow \mathtt{N}, f(s) \models \phi$ *holds for all* $s \in S$ *and for all formulas* ϕ.
(ii) f *is* $\mathcal{INV}(\mathcal{B}(S), \lambda_{\mathtt{M}})$-$\mathcal{INV}(\mathcal{B}(T), \lambda_{\mathtt{N}})$-*measurable.*

Proof. (1) Part (i) is established through an easy induction on formula ϕ.

(2) Part (ii) uses part (i), because the latter implies for each formula $\phi \in \mathbf{F}_{\mathcal{L}}$ the equality $f^{-1}[[\![\phi]\!]_{\mathtt{N}}] = [\![\phi]\!]_{\mathtt{M}}$. Consider now

$$\mathcal{G} := \{D \in \mathcal{INV}(\mathcal{B}(T), \lambda_{\mathtt{N}}) \mid f^{-1}[D] \in \mathcal{INV}(\mathcal{B}(S), \lambda_{\mathtt{M}})\},$$

then $\mathcal{G}$ is a σ-algebra on T which contains all the elements $[\![\phi]\!]_{\mathtt{N}}$ of a generator of $\mathcal{INV}(\mathcal{B}(T), \lambda_{\mathtt{N}})$. Thus $\mathcal{G} = \mathcal{INV}(\mathcal{B}(T), \lambda_{\mathtt{N}})$. $\square$

This observation is adapted to the situation at hand by introducing randomized morphisms that cooperate with the logic.

Definition 5.4. Let $\mathtt{M}$ and $\mathtt{N}$ be Markov transition systems over the Polish state spaces S resp. T. Morphism $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ is called a $\mathcal{L}$-*morphism* iff

$$\Phi : (S, \mathcal{INV}(\mathcal{B}(S), \lambda_{\mathtt{M}})) \rightsquigarrow (T, \mathcal{INV}(\mathcal{B}(T), \lambda_{\mathtt{N}})).$$

Thus we construct $\mathcal{L}$-morphisms in analogy to the case of non-randomized morphisms between transition systems in the base category. If Φ is a $\mathcal{L}$-morphism, then the set

$$\{s \in S \mid \Phi(s)([\![\phi]\!]_{\mathtt{M}}) \geq q\}$$

is $\mathcal{INV}(\mathcal{B}(S), \lambda_{\mathtt{M}})$-invariant for each formula ϕ, hence the elements of this set cannot be separated by the logic. This means that $\Phi(s)([\![\phi]\!]_{\mathtt{M}}) \geq q$ iff $\Phi(s')([\![\phi]\!]_{\mathtt{M}}) \geq q$, provided s and s' satisfy the same formulas in $\mathcal{L}$. Consequently (see the proof of Lemma 5.1),

$$\Phi(s)([\![\phi]\!]_{\mathtt{M}}) = \Phi(s')([\![\phi]\!]_{\mathtt{M}})$$

holds for all formulas ϕ, whenever $s\ \lambda_{\mathtt{M}}\ s'$, and vice versa. To be more specific, a characterization of these morphisms in terms of congruences reads:

Proposition 5.5. *Let* $\mathtt{M}$ *and* $\mathtt{N}$ *be Markov transition systems. Then the randomized morphism* $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ *is a* $\mathcal{L}$-*morphism iff* $(\lambda_{\mathtt{M}}, \lambda_{\mathtt{N}})$ *is a congruence for* Φ.

Proof. This follows immediately from Lemma 2.8. □

$\mathcal{L}$-morphisms can be used to build up a category (which will not be done here). Just for illustrating the concept, we mention

Corollary 5.6. *Let* $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ *and* $\Psi : \mathtt{N} \curvearrowright \mathtt{P}$ *be randomized morphisms, then*

(i) *if both* Φ *and* Ψ *are* $\mathcal{L}$-*morphisms, so is* $\Psi * \Phi : \mathtt{M} \curvearrowright \mathtt{P}$,
(ii) *if* $(\lambda_{\mathtt{M}}, \lambda_{\mathtt{N}})$ *is a congruence for* Φ, *and* $(\lambda_{\mathtt{N}}, \lambda_{\mathtt{P}})$ *is a congruence for* Ψ, *then* $(\lambda_{\mathtt{M}}, \lambda_{\mathtt{P}})$ *is a congruence for the Kleisli product* $\Psi * \Phi$.

Proof. Property (i) follows from the fact that the Kleisli composition of Kleisli morphisms is again a Kleisli morphism. Property (ii) is a consequence of (i), using Proposition 5.5. □

Corollary 5.7. *Let* $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ *be a* $\mathcal{L}$-*morphism, then* $\mu\ \overline{\lambda_{\mathtt{M}}}\ \mu'$ *implies* $\overline{\Phi}(\mu)\ \overline{\lambda_{\mathtt{N}}}\ \overline{\Phi}(\mu')$.

Proof. This is immediate from Lemma 4.6. □

Call a smooth equivalence relation τ on the state space S of the Markov transition system $\mathtt{M} = (S, (k_a)_{a \in \mathsf{A}})$ a $\mathtt{M}$-*congruence* iff τ is a congruence for each $k_a : S \rightsquigarrow S$. The factor system

$$\mathtt{M}/\tau := \big(S/\tau, (k_{a,\tau})_{a \in \mathsf{A}}\big)$$

has as a state space the analytic space S/τ of τ-equivalence classes with the transition rules $k_{a,\tau} : S/\tau \rightsquigarrow S/\tau$ for each action $a \in \mathsf{A}$; see Proposition 2.9 for factoring stochastic relations.

Define for $\mathtt{M}$ the $\mathcal{L}$-*reduced model* $\mathtt{M}/\mathcal{L}$ as $\mathtt{M}/\lambda_{\mathtt{M}}$, so that two different states in $\mathtt{M}/\mathcal{L}$ can always be separated by the logic. We show that the reduction does not destroy the property of being a $\mathcal{L}$-morphism (modulo factoring, of course).

Before doing that, we briefly investigate the reduced model with respect to non-randomized morphisms. Let $f : \mathtt{M} \to \mathtt{N}$ be a morphism, then $\mathsf{ker}(f) \subseteq \lambda_{\mathtt{M}}$, because $f(s) = f(s')$ implies $\mathtt{N}, f(s) \models \phi$ iff $\mathtt{N}, f(s') \models \phi$, and since $\mathtt{M}, s \models \phi$ iff $\mathtt{N}, f(s) \models \phi$ by Lemma 5.3, (i), we see that $f(s) = f(s')$ entails $\langle s, s'\rangle \in \lambda_{\mathtt{M}}$. It is clear that $\mathsf{ker}(f)$ is a $\mathtt{M}$-congruence. These observations yield a characterization of the reduced model.

Proposition 5.8. *Let* $f : \mathtt{M} \to \mathtt{N}$ *be a morphism, then there exists a unique model morphism* $\varpi_{f,\mathcal{L}} : \mathtt{M}/\mathsf{ker}(f) \to \mathtt{M}/\mathcal{L}$ *with* $\eta_{\mathcal{L}} = \varpi_{f,\mathcal{L}} \circ \eta_{\mathsf{ker}(f)}$.

Proof. The claim follows directly from $\mathsf{ker}(f) \subseteq \lambda_{\mathtt{M}}$, and from parts (iii) and (iv) of Proposition 2.9. This is so because the morphism constructed in the latter part does only depend on the congruences and not on the specific stochastic relations that are involved. □

Returning to the discussion of $\mathcal{L}$-morphisms, we show now that a such a morphism between two models gives rise to a morphism between the reduced models.

Proposition 5.9. *Let* $\mathtt{M}$ *and* $\mathtt{N}$ *be Markov transition systems with state spaces* S *resp.* T, *and assume that* $\Phi : \mathtt{M} \curvearrowright \mathtt{N}$ *is a* $\mathcal{L}$-*morphism. Define* $\Phi_{\mathcal{L}} : S/\lambda_{\mathtt{M}} \rightsquigarrow T/\lambda_{\mathtt{N}}$ *through* $\Phi_{\mathcal{L}} := \Phi_{(\lambda_{\mathtt{M}}, \lambda_{\mathtt{N}})}$, *the factor relation induced by the congruence* $(\lambda_{\mathtt{M}}, \lambda_{\mathtt{N}})$. *Then*

$$\Phi_{\mathcal{L}} : \mathtt{M}/\mathcal{L} \curvearrowright \mathtt{N}/\mathcal{L}$$

is a $\mathcal{L}$-*morphism.*

Proof. (1) We show first that $\Phi_{\mathcal{L}}$ is a randomized morphism $\mathtt{M}/\mathcal{L} \curvearrowright \mathtt{N}/\mathcal{L}$. Let $k_a : S \rightsquigarrow S$ and $\ell_a : T \rightsquigarrow T$ be the respective transition laws for action $a \in \mathsf{A}$. Take $s \in S$ and $G \in \mathcal{B}(T/\lambda_{\mathtt{N}})$. Then

$$\begin{aligned}
\left(\ell_{a,\lambda_{\mathtt{N}}} * \Phi_{\mathcal{L}}\right)([s]_{\lambda_{\mathtt{M}}})(G) &= \int_{T/\lambda_{\mathtt{N}}} \ell_{a,\lambda_{\mathtt{N}}}(y)(G)\ \Phi_{\mathcal{L}}([s]_{\lambda_{\mathtt{M}}})(\mathrm{d}y) \\
&\overset{\text{(CV)}}{=} \int_T \ell_a(t)(\eta_{\lambda_{\mathtt{N}}}^{-1}[G])\ \Phi(s)(\mathrm{d}t) \\
&\overset{\text{(RM)}}{=} \int_S \Phi(w)(\eta_{\lambda_{\mathtt{N}}}^{-1}[G])\ k_a(s)(\mathrm{d}w) \\
&= \int_S \Phi_{\mathcal{L}}([w]_{\lambda_{\mathtt{N}}})(G)\ k_a(s)(\mathrm{d}w) \\
&\overset{\text{(CV)}}{=} \int_{S/\lambda_{\mathtt{M}}} \Phi_{\mathcal{L}}(v)(G)\ k_{a,\lambda_{\mathtt{M}}}([s]_{\lambda_{\mathtt{M}}})(\mathrm{d}v) \\
&= \left(\Phi_{\mathcal{L}} * k_{a,\lambda_{\mathtt{M}}}\right)([s]_{\lambda_{\mathtt{M}}})(G).
\end{aligned}$$

The equalities marked (CV) use the Change of Variables formula, the equality marked (RM) derives from Φ being a randomized morphism.

Consequently, $\Phi_{\mathcal{L}}$ is a randomized morphism $\mathtt{M}/\mathcal{L} \curvearrowright \mathtt{N}/\mathcal{L}$.

(2) Because $\lambda_{\mathtt{M}/\mathcal{L}}$ equals the identity $\Delta_{S/\mathcal{L}}$ on $S/\mathcal{L}$, Example 3.3, part (i) tells us that

$$\mathcal{INV}(\mathcal{B}(S/\lambda_{\mathtt{M}}), \lambda_{\mathtt{M}/\mathcal{L}}) = \mathcal{B}(S/\lambda_{\mathtt{M}}).$$

Similarly for $\mathtt{N}$. Thus

$$\Phi_{\mathcal{L}} : \mathcal{INV}(\mathcal{B}(S/\lambda_{\mathtt{M}}), \lambda_{\mathtt{M}/\mathcal{L}}) \rightsquigarrow \mathcal{INV}(\mathcal{B}(T/\lambda_{\mathtt{N}}), \lambda_{\mathtt{N}/\mathcal{L}}),$$

is a stochastic relation. Consequently, $\Phi_{\mathcal{L}}$ is a $\mathcal{L}$-morphism. $\square$

6. Factoring through a randomized congruence

When showing that congruences are really the kernels of morphisms, one readily proceeds to discuss factoring. A morphism in universal (co-) algebra can be factored uniquely through the factor space associated with its kernel. This is also the case for stochastic relations [8, Section 5.2], and it helps for example to identify simple objects.

We will look into this problem by investigating first the factor of the Kleisli extension of a stochastic relation with respect to a general congruence. Alas, this is not automatically the Kleisli extension of a factored relation, but if the relation is grounded, the isomorphism from Section 3.1 comes in helpfully and renders this map a factor, at least up to a Borel isomorphism. In general, we show that there is a one-to-one correspondence of near-grounded randomized congruences and the kernels of near-grounded randomized morphisms. This relationship is investigated more closely.

Let $\mathsf{K} = (X, Y, K)$ be an analytic stochastic relation, and assume that $\mathsf{D} = (\rho, \tau)$ is a randomized congruence for K. Define

$$\overline{K}/\mathsf{D} : \begin{cases} \mathbb{S}(X)/\rho \to \mathbb{S}(Y)/\tau \\ [\mu]_\rho \mapsto \left[\overline{K}(\mu)\right]_\tau . \end{cases}$$

It is clear from Lemma 3.8 that the map $\overline{K}/\mathsf{D}$ is well defined, and that it constitutes a measurable map. $\overline{K}/\mathsf{D}$ is defined as the canonical map on factors that renders

$$\begin{tikzcd}
\mathbb{S}(X) \arrow[r, "\overline{K}"] \arrow[d, "\eta_\rho"'] & \mathbb{S}(Y) \arrow[d, "\eta_\tau"] \\
\mathbb{S}(X)/\rho \arrow[r, "\overline{K}/\mathsf{D}"'] & \mathbb{S}(Y)/\tau
\end{tikzcd}$$

commutative. This diagram is a diagram of maps between sets, and it is not obvious whether or not it is the Kleisli extension of a stochastic relation at all. Consider on the other hand the stochastic relation

$$\mathsf{K}/\lfloor \mathsf{D} \rfloor := (X/\lfloor \rho \rfloor, Y/\lfloor \tau \rfloor, K/\lfloor \mathsf{D} \rfloor).$$

Then the diagram below is obtained. Note that by Lemma 3.8 ∂_ρ is a Borel map because D is near-grounded.

$$\begin{tikzcd}
\mathbb{S}(X)/\rho \arrow[r, "\partial_\rho"] \arrow[d, "\overline{K}/\mathsf{D}"'] \arrow[dr, phantom, "(\bigstar)"] & \mathbb{S}(X/\lfloor \rho \rfloor) \arrow[d, "\overline{K/\lfloor \mathsf{D} \rfloor}"] \\
\mathbb{S}(Y)/\tau \arrow[r, "\partial_\tau"'] & \mathbb{S}(Y/\lfloor \tau \rfloor)
\end{tikzcd}$$

The diagram commutes: take $\mu \in \mathbb{S}(X)$ and a Borel set $B \in \mathcal{B}(Y/\lfloor \tau \rfloor)$ then

$$\overline{K/\lfloor \mathsf{D} \rfloor}\left(\partial_\rho([\mu]_\rho)\right)(B) = \int_{X/\lfloor \rho \rfloor} (K/\lfloor \mathsf{D} \rfloor)(\zeta)(B)\, \partial_\rho([\mu]_\rho)(\mathrm{d}\zeta) \tag{12}$$

$$= \int_{X/\lfloor \rho \rfloor} (K/\lfloor \mathsf{D} \rfloor)(\zeta)(B)\, \mathbb{S}(\eta_{\lfloor \rho \rfloor})(\mu)(\mathrm{d}\zeta) \tag{13}$$

$$= \int_X (K/\lfloor \mathsf{D} \rfloor)([x]_{\lfloor \rho \rfloor})(B)\, \mu(\mathrm{d}x) \tag{14}$$

$$= \int_X K(x)(\eta_{\lfloor \tau \rfloor}^{-1}[B])\, \mu(\mathrm{d}x) \tag{15}$$

$$= \overline{K}(\mu)(\eta_{\lfloor \tau \rfloor}^{-1}[B]) \tag{16}$$

$$= \partial_\tau\left(\left[\overline{K}(\mu)\right]_\tau\right)(B) \tag{17}$$

$$= \partial_\tau\left((\overline{K}/\mathsf{D})([\mu]_\rho)\right)(B). \tag{18}$$

Eq. (12) is the definition of $K/\lfloor \mathsf{D} \rfloor \mapsto \overline{K/\lfloor \mathsf{D} \rfloor}$, Eq. (13) expands the definition of map ∂_ρ, the next Eq. (14) applies the Change of Variables formula. The last group of Eqs. (16)–(18) uses the definition of map ∂_τ and the construction of the factor map $\overline{K}/\mathsf{D}$ again.

Associate with the randomized congruence $\mathsf{D} = (\rho, \tau)$ the maps

$$E_\rho : X \ni x \mapsto \mathfrak{e}_{X/\lfloor \rho \rfloor}([x]_{\lfloor \rho \rfloor}) \in \mathbb{S}(X/\lfloor \rho \rfloor),$$

$$E_\tau : Y \ni y \mapsto \mathfrak{e}_{Y/\lfloor \tau \rfloor}([y]_{\lfloor \tau \rfloor}) \in \mathbb{S}(Y/\lfloor \tau \rfloor).$$

Thus $E_\rho = \mathfrak{e}_{X/\lfloor \rho \rfloor} \circ \eta_{\lfloor \rho \rfloor}$, $E_\tau = \mathfrak{e}_{Y/\lfloor \tau \rfloor} \circ \eta_{\lfloor \tau \rfloor}$. Define $\mathsf{E}_\mathsf{D} := (E_\rho, E_\tau)$.

If D is grounded, both components (ρ, τ) are, so that in this case both ∂_ρ and ∂_τ are Borel isomorphisms by Proposition 3.11. Thus $\overline{K}/\mathsf{D}$ is the Kleisli map of a stochastic relation up to a Borel isomorphism, provided D is a grounded congruence.

Proposition 6.1. *Let D be a congruence for the stochastic relation K. Then*

(i) $\mathsf{E}_\mathsf{D} : \mathsf{K} \curvearrowright \mathsf{K}/\lfloor \mathsf{D} \rfloor$ *is a randomized morphism with* $\ker(\mathsf{E}_\mathsf{D}) = \lfloor \mathsf{D} \rfloor$.
(ii) *If D is grounded, the Kleisli extension $\overline{\mathsf{K}/\lfloor \mathsf{D} \rfloor}$ of $\mathsf{K}/\lfloor \mathsf{D} \rfloor$ is Borel isomorphic to the factor $\overline{\mathsf{K}}/\mathsf{D}$ of the Kleisli extension of K with respect to D.*

Proof. (1) We infer from Corollary 4.8 that $\lfloor \mathsf{D} \rfloor$ is a congruence for K. This implies that $(\eta_{\lfloor \rho \rfloor}, \eta_{\lfloor \tau \rfloor}) : \mathsf{K} \to \mathsf{K}/\lfloor \mathsf{D} \rfloor$ is a morphism. Lemma 4.4 shows now that E_D is a randomized morphism.

(2) If D is grounded, diagram ($\bigstar$) above gives

$$\overline{\mathsf{K}/\lfloor\mathsf{D}\rfloor} = \partial_\rho^{-1} \circ \overline{\mathsf{K}}/\mathsf{D} \circ \partial_\tau$$

for the Borel isomorphisms ∂_ρ and ∂_τ. □

Now let $\mathsf{F} = (\Phi, \Psi) : \mathsf{K} \curvearrowright \mathsf{L}$ be a near-grounded randomized morphism, then we conclude from Lemma 4.10 and from Proposition 4.13 that $\mathsf{ker}(\overline{\mathsf{F}})$ is a randomized congruence for K, and we know that $\lfloor \mathsf{ker}(\overline{\mathsf{F}}) \rfloor = \mathsf{ker}(\mathsf{F})$. Thus we obtain a randomized morphism

$$\mathsf{E}_{\mathsf{ker}(\overline{\mathsf{F}})} : \mathsf{K} \curvearrowright \mathsf{K}/\mathsf{ker}(\mathsf{F})$$

from the construction in Proposition 6.1. We can say a wee bit more.

Proposition 6.2. *Let $\mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$ be a near-grounded randomized morphism, then there exists a unique morphism $\mathsf{G} : \mathsf{K}/\mathsf{ker}(\mathsf{F}) \curvearrowright \mathsf{L}$ that makes the diagram*

$$\begin{tikzcd}
\mathsf{K} \arrow[r, dashed, "\mathsf{F}"] \arrow[d, dashed, "\mathsf{E}_{\mathsf{ker}(\overline{\mathsf{F}})}"'] & \mathsf{L} \\
\mathsf{K}/\mathsf{ker}\,(\mathsf{F}) \arrow[ur, dashed, "\mathsf{G}"'] &
\end{tikzcd}$$

commutative.

Proof. (1) Assume $\mathsf{F} = (\Phi, \Psi)$ with $\Phi : X \rightsquigarrow A$ and $\Psi : Y \rightsquigarrow B$. Put

$$\Gamma([x]_{\mathsf{ker}(\Phi)}) := \Phi(x),$$
$$\Theta([y]_{\mathsf{ker}(\Psi)}) := \Psi(y).$$

Then evidently $\Gamma : X/\mathsf{ker}(\Phi) \rightsquigarrow A$ and $\Theta : Y/\mathsf{ker}(\Psi) \rightsquigarrow B$, and it is apparent that both $\overline{\Gamma}$ and $\overline{\Theta}$ are onto. Thus we have to show that $\mathsf{G} := (\Gamma, \Theta)$ is a randomized morphism $\mathsf{G} : \mathsf{K}/\mathsf{ker}(\mathsf{F}) \curvearrowright \mathsf{L}$ that makes the diagram above commute.

(2) G is a randomized morphism: let $[x]_{\mathsf{ker}(\Phi)} \in X/\mathsf{ker}(\Phi)$ and $D \in \mathcal{B}(B)$, then we obtain on account of $\mathsf{F} = (\Phi, \Psi)$ being a randomized morphism

$$\begin{aligned}
(L * \Gamma)\,([x]_{\mathsf{ker}(\Phi)})(D) &= \int_A L(a)(D)\ \Gamma([x]_{\mathsf{ker}(\Phi)})(\mathrm{d}a) \\
&= \int_A L(a)(D)\ \Phi(x)(\mathrm{d}a) \\
&= \int_Y \Psi(y)(D)\ K(x)(\mathrm{d}y) \\
&\overset{(\dagger)}{=} \int_{Y/\mathsf{ker}(\Psi)} \Theta(t)(D)\ \ (\mathsf{K}/\mathsf{ker}(\mathsf{F}))\,([x]_{\mathsf{ker}(\Phi)})(\mathrm{d}t) \\
&= (\Theta * \mathsf{K}/\mathsf{ker}(\mathsf{F}))\,([x]_{\mathsf{ker}(\Phi)})(D).
\end{aligned}$$

Equation (†) follows from the Change of Variables formula. Thus we have shown that $L * \Gamma = \Theta * \mathsf{K}/\mathsf{ker}(\mathsf{F})$ holds, which implies that G is in fact a randomized morphism.

(3) Now let again $D \in \mathcal{B}(A)$ be a Borel set, and $x \in X$. Then

$$\left(\Gamma * E_{\mathsf{ker}(\Phi)}\right)(x)(D) = \int_{X/\mathsf{ker}(\Phi)} \Gamma(s)(D)\ E_{\mathsf{ker}(\Phi)}(x)(\mathrm{d}s) = \Gamma([x]_{\mathsf{ker}(\Phi)})(D) = \Phi(x)(D).$$

Thus $\Phi = \Gamma * E_{\mathsf{ker}(\Phi)}$, and Γ is uniquely determined. Similarly one shows that $\Psi = \Theta * E_{\mathsf{ker}(\Psi)}$, and that Θ is unique. But this means $\mathsf{F} = \mathsf{G} * \mathsf{E}_{\mathsf{ker}(\overline{\mathsf{F}})}$, as claimed. □

Now we are in a position to relate the kernels of near-grounded morphisms to near-grounded congruences.

Corollary 6.3. *These conditions are equivalent for a congruence* E *for stochastic relation* K

(i) $\mathsf{E} = \mathsf{ker}(\mathsf{F})$ *for a near-grounded morphism* $\mathsf{F} : \mathsf{K} \curvearrowright \mathsf{L}$.
(ii) $\mathsf{E} = \lfloor \mathsf{D} \rfloor$ *for a randomized congruence* D *for* K.

Proof. The implication '(i) $\Rightarrow$ (ii)' follows from Proposition 4.13, '(ii) $\Rightarrow$ (i)' is an immediate consequence of Proposition 6.1, part (i). $\square$

A stronger observation has been made in the non-randomized case. Here it could be shown that for the kernels of morphisms are exactly the congruences. The case of randomized congruences and morphisms turns out to be considerably more involved and colorful.

7. Conclusion and further work

Stochastic relations are the Kleisli morphisms for the Giry monad, and this paper proposes the study of the associated morphisms and congruences with the algebraically oriented goal of investigating the relationship between both: it is established that the kernel of a morphism is a congruence, and vice versa, and a unique factorization of a morphism through this kernel can be found.

Specifically, this paper contributes to the study of randomized morphisms, i.e., Kleisli morphisms for the Giry monad, and to randomized congruences, i.e. congruences on the space of subprobabilities, in the following way.

- Countably generated equivalence relations on the space $\mathbb{S}(X)$ for analytic spaces X are studied. These relations may sometimes be defined by lifting an equivalence relation from X to $\mathbb{S}(X)$; conversely, equivalence relations on X may be traced from those on $\mathbb{S}(X)$ through the unit of the Giry monad. Tracing and lifting is investigated quite closely using positive convex structures, indicating cross connections to Eilenberg–Moore algebras for this monad.
- Randomized morphisms for stochastic relations are defined, and the class of near-grounded relations is studied closely. Using transportation through Borel maps, we refine a reduction technique from analytic spaces to Polish ones (where life is sometimes a bit easier).
- Randomized morphisms are defined as morphisms in the Kleisli category for the Giry monad, their kernels are studied, and it is shown that near-grounded morphisms have near-grounded kernels. This is used for the factorization of a randomized morphism through its kernel.
- As an illustration, we look briefly at Hennessy–Milner logic and define randomized morphism for their models. It is then discussed under which conditions the logic defines congruences for these morphisms, emulating the behavior under non-randomized morphisms.

Further work. This paper is perceived as a first step towards the study of morphisms in the Kleisli category associated with the Giry monad. Previous work on stochastic relations has indicated that the subprobability functor has some properties which suggest that coalgebraic properties and techniques require substantial modifications when discussed in the context of this functor and the associated coalgebras. This seems to be the case for the Kleisli category as well. Hence a host of questions needs to be asked. For example, the definition of bisimulations and their relationship to simple systems requires close scrutiny. Going a step further, the question of the existence of various helpful constructs such as (semi-) pullbacks needs to be answered. It is also certainly helpful to enter the study of non-randomized morphisms for stochastic Kripke models.

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